Solving the RBC Model *

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1 Basic Neoclassical Growth Model without Labor Supply

We study how to solve a RBC model using Dynare in this note. We work with a neoclassical growth model without labor supply as a baseline throughout the note.

Definition. Recursive competitive equilibrium is the list $R(z,K)$, $W(z,K)$, $V(z,K,k)$, $g_k(z,K,k)$, $g_\ell(z,K,k)$, $G_L(z,K)$ and $G_K(z,K)$, such that:

1. The representative consumer solves the following problem with $V(z,K,k)$. The associated optimal decision rule is $g_k(z,K,k)$ and $g_\ell(z,K,k)$ for savings and labor supply. In particular, the recursive formulation of the consumer's problem is as follows:

$$V(z,K,k) = \max_{k'\geq 0,\ \ell\in[0,1]} \log c + \beta E_{z'|z} V\left(z',K',k'\right)$$
$$\text{s.t. } c + k' = (R(z,K)+1-\delta)k + W(z,K)\ell.$$

2. The representative firm maximizes its profit as follows:

$$\max_{K^d,L^d} = F(L^d,K^d) - R(z,K)K^d - W(z,K)L^d$$
$$\text{s.t. } F(K) = e^z L^{1-\alpha} K^\alpha$$
$$z' = \rho z + \epsilon, \quad \epsilon \sim N\left(0,\sigma_\epsilon^2\right) \text{ iid.}$$

3. Optimal decision rule of consumers is consistent with the law of motion of aggregate capital and labor:

$$K' = G_K(z,K) = g_k(z,K,K)$$
$$L = G_L(z,K) = g_\ell(z,K,K)$$

*This is an abridged note of *Solving RBC models with Linearized Euler Equations: Blanchard-Kahn Method* by Makoto Nakajima. Refer to the original material for more details.

Solving for the household problem, we first get $g_\ell(z, K, k) = 1$ because households would want to supply as much labor as they can because there is no labor disutility. The first-order condition with respect to k' and the envelope condition yield the following:

$$\frac{1}{c} + \beta E_{z'|z} V_k(z', K', k') = 0$$
$$V_k(z, K, k) = \frac{R(z, K) + 1 - \delta}{c}.$$

Combining these two equations, we obtain:

$$\frac{1}{C} = \beta E_{z'|z} \frac{R' + 1 - \delta}{C'}, \tag{1}$$

where C denotes aggregate consumption.

Also, a representative firm's optimal condition with respect to K^d and L^d yields the following:

$$R = \alpha e^z L^{1-\alpha} K^{\alpha-1} \tag{2}$$
$$W = (1-\alpha) e^z L^{-\alpha} K^{\alpha} \tag{3}$$

Equations (1), (2), and (3), together with the household's budget constraint and law of motion of z, constitute the equilibrium of this economy. Hence, the equilibrium consists of 5 difference equations of four endogenous variables C, K', R, W, an exogenous variable z. Each period, the state variables are K and z and the control variables that are endogenously chosen within the period are C, K', R, and W.

$$\frac{1}{C} = \beta E_{z'|z} \frac{R' + 1 - \delta}{C'} \tag{4}$$
$$R = \alpha e^z K^{\alpha-1} \tag{5}$$
$$W = (1-\alpha) e^z K^{\alpha} \tag{6}$$
$$C + K' = (R + 1 - \delta)K + W \tag{7}$$
$$z' = \rho z + \epsilon \tag{8}$$

2 Steady State and Log-linearization

We need to solve for the steady state equilibrium before we take the first-order approximation of the nonlinear difference equations obtained above. In the steady state, the exogenous variable z is at its unconditional mean level, which is 0, and the endogenous variables are constant.

$$\frac{1}{\bar{C}} = \beta \frac{\bar{R} + 1 - \delta}{\bar{C}}$$
$$\bar{C} + \bar{K} = (\bar{R} + 1 - \delta)\bar{K} + \bar{W}$$
$$\bar{R} = \alpha \bar{K}^{\alpha-1}$$
$$\bar{W} = (1-\alpha)\bar{K}^{\alpha}$$

Thus, we get the following.

$$\bar{R} = \frac{1}{\beta} - 1 + \delta$$
$$\bar{C} = \bar{K}^{\alpha} - \delta \bar{K}$$
$$\bar{W} = (1-\alpha)\bar{K}^{\alpha}$$
$$\bar{K} = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{\frac{1}{\alpha-1}}$$

Now, we linearize the difference equations (4), (5), (6), and (7) around the steady state by using first-order Taylor approximation: $F(X) = F(\bar{X}) + F'(\bar{X})(X - \bar{X})$. We denote $x = \frac{X-\bar{X}}{\bar{X}}$ henceforth. First, we can linearize (4) as follows:

$$\begin{aligned}
&(4): \ \frac{1}{\bar{C}} - \frac{C-\bar{C}}{\bar{C}^2} = \frac{1}{\bar{C}} + \frac{\beta}{\bar{C}} E_{z'|z}(R' - \bar{R}) - \frac{1}{\bar{C}^2} E_{z'|z}(C' - \bar{C}) \\
&\iff -c = -E_{z'|z}c' + \beta\left(\frac{1}{\beta} - 1 + \delta\right) E_{z'|z} r'
\end{aligned}$$

Second, firms' optimal conditions (5) and (6) are linearized as follows:

$$\begin{aligned}
&(5): \ \bar{R} + (R - \bar{R}) = \alpha \bar{K}^{\alpha-1} + \alpha z \bar{K}^{\alpha-1} + \alpha(\alpha-1)\bar{K}^{\alpha-2}(K - \bar{K}) \\
&\iff r = z + (\alpha - 1)k \\
&(6): \ \bar{W} + (W - \bar{W}) = (1-\alpha)\bar{K}^{\alpha} + (1-\alpha)\bar{K}^{\alpha} z + \alpha(1-\alpha)\bar{K}^{\alpha-1}(K - \bar{K}) \\
&\iff w = z + \alpha k
\end{aligned}$$

Third, household budget constraint can be linearized as follows:

$$\begin{aligned}
&(7): \ \bar{C} + \bar{K} + (C - \bar{C}) + (K' - \bar{K}) = (\bar{R} + 1 - \delta)\bar{K} + \bar{W} + \bar{K}(R - \bar{R}) + (\bar{R} + 1 - \delta)(K - \bar{K}) + (W - \bar{W}) \\
&\iff \bar{C}c + \bar{K}k' = \bar{K}\bar{R}r + (\bar{R} + 1 - \delta)\bar{K}k + \bar{W}w
\end{aligned}$$

Lastly, we have $z' = \rho z + \epsilon$ for the exogenous variable z. These five equations constitute the linearized equilibrium.

3 Solving the Linearized Model

The linearized equations can be written in a matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \bar{K} & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 + \beta(1-\delta) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z' \\ k' \\ Ec' \\ Er' \\ Ew' \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & (\bar{R} + 1 - \delta)\bar{K} & -\bar{C} & \bar{K}\bar{R} & \bar{W} \\ 0 & 0 & 1 & 0 & 0 \\ 1 & \alpha - 1 & 0 & -1 & 0 \\ 1 & \alpha & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z \\ k \\ c \\ r \\ w \end{bmatrix} + \begin{bmatrix} \epsilon \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

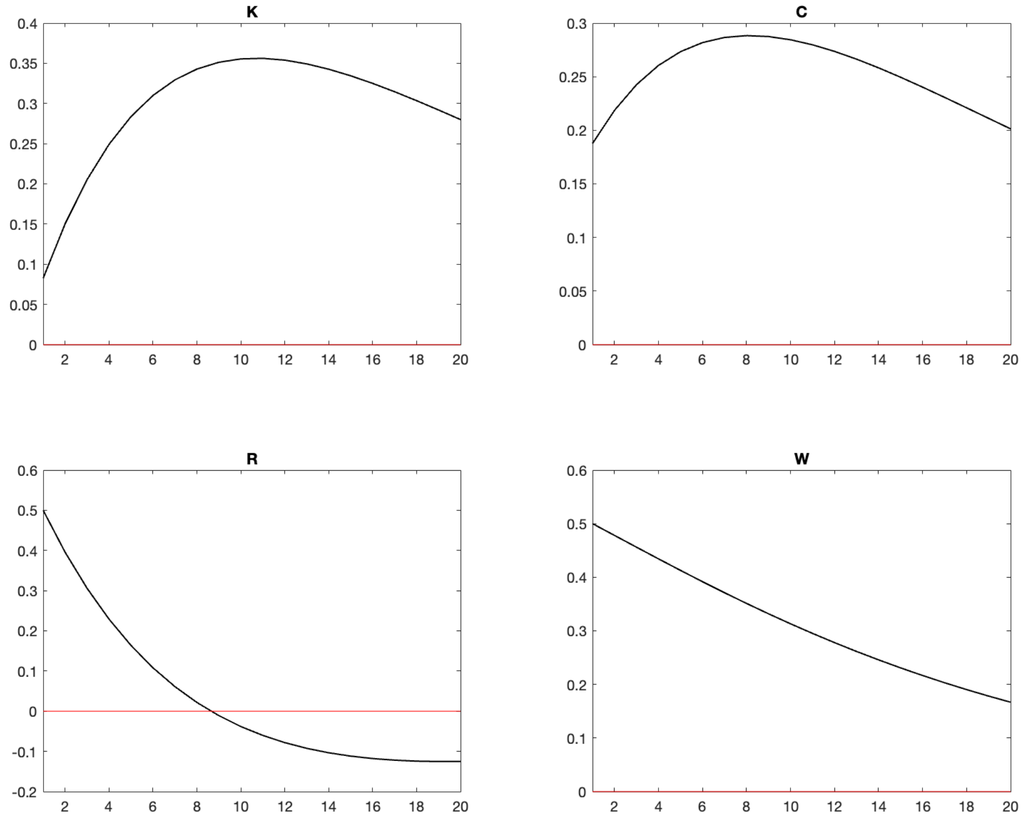

Figure 1: Impulse Responses to 1-SD Shock to TFP

We can rewrite the matrix representation above as follows:

$$A \begin{bmatrix} x' \\ Ey' \end{bmatrix} = B \begin{bmatrix} x \\ y \end{bmatrix} + C\nu'$$

Denote the number of control variables $y = \{c, r, w\}$ as m. Also, define $F = A^{-1}B$ and let h be the number of eigenvalues of F outside the unit circle. Then Blanchard-Kahn condition states when the solution of the system is unique.

Proposition. (Blanchard-Kahn condition) If $h = m$, the solution of the system is unique. If $h > m$, there is no solution to the system. If $h < m$, there is an infinite number of solutions.

We abstract from fully specifying the solution to this linear system. However, we can solve the model using Dynare. You can put in either the nonlinear version or the linear version of the model in Dynare and check if the model satisfies the Blanchard-Kahn condition. This model satisfies the Blanchard-Kahn condition, and we can compute impulse responses of capital, consumption, interest rate, and wage given a 1 standard deviation shock to TFP z.

Figure 1 depicts responses of K, C, R, and W given a TFP shock measured as percentage deviation from steady state values in the nonlinear model. We get exactly the same figure from the linearized model. When there is a positive shock to TFP, this increases output today so both capital stock and consumption increase over time. Also, both interest rate and wage go up in this model because marginal product of both capital and labor go up with higher TFP. However, R goes down below the steady state level as capital stock is abundant while the TFP shock diminishes away.

Last thing to remark is that you have to be careful with the units of impulse responses. Figure 1 shows that capital rises by slightly less than 10% of the steady state value, which is easy to interpret. However, interest rate rises by almost 50% of the steady state level, and this is not the usual way we illustrate changes in interest rates. The interest rate goes up by 4 percentage points in the nonlinear model. Sometimes, we use percentage-point changes from the steady state level when analyzing variables like interest rates or inflation rates. Hence, the best way to do this is save the equilibrium object that Dynare throws out and compute impulse-responses in an appropriate unit for each variable by yourself.