ECON 714A: Macroeconomic Theory - Handout 1

Professor Job Boerma

TA: Yeonggyu Yun

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Class Logistics

- TA sessions: Thursday 11:00-12:15PM at Grainger Hall 2120
- TA e-mail address: yyun23@wisc.edu
- TA office hours: Wednesday 3:30-5PM at Social Science 6439
- Mid-term Exam: March 6th, 11:00-12:15PM in class
- HW submission: Please submit one per each group (4-5 people) in a printed form at my TA mailbox in front of TA resource room 7218 by Friday 5PM.

Outline of SLP Chapter 3

- We solve a dynamic programming problem as (1) by updating the function v iteratively. We construct a sequence of functions $\{v_n\}$. We want to show that $v_n \to v$ for any choice of v_0.

$$v(k) = \max_{0 \le y \le f(k)} U(f(k) - y) + \beta v(y) \tag{1}$$

$$\Rightarrow v_{n+1}(k) = \max_{0 \le y \le f(k)} U(f(k) - y) + \beta v_n(y)$$

- That is technically showing the existence and uniqueness of v. In the process, we construct an operator $T : C \to C$ defined as below. Finding solution v for (1) is equivalent to finding a fixed point of the mapping T.

$$Tv(k) = \max_{0 \le y \le f(k)} U(f(k) - y) + \beta v(y)$$

- SLP Chapter is organized as follows: Section 3.1 tells us the properties of the space C of interest and Section 3.2 introduces the Contraction Mapping Theorem, which is the fixed point theorem in this setting. Section 3.3 tells us more about the properties of T in relation to maximization.

SLP 3.1. Metric Spaces and Normed Vector Spaces

Definition. A vector space X is a set of vectors with two operations, addition and scalar multiplication. For any two vectors $x, y \in X$ and $\alpha \in \mathbb{R}$, we have $x + y \in X$ and $\alpha x \in X$. The operations also follow the algebraic rules. For all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$:

(a) $x + y = y + x$

(b) $(x + y) + z = x + (y + z)$

(c) $\alpha(x + y) = \alpha x + \alpha y$

(d) $(\alpha + \beta)x = \alpha x + \beta x$

(e) $(\alpha\beta)x = \alpha(\beta x)$

(f) $\exists\, \theta \in X$ s.t. $x + \theta = x;\ 0x = \theta$; We call θ zero vector.

(g) $1x = x$

Exercise 3.2.d. Show that the set of all continuous functions on the interval $[a, b]$ is a vector space.
Solution: For any two real-valued continuous functions on $[a, b]$, f and g, we define $(f + g)(x) = f(x)+g(x)$ for $x \in [a, b]$ and $cf(x) = c \cdot f(x)$ for $x \in [a, b]$ for a constant $c \in \mathbb{R}$. Then for any $x \in [a, b]$, any $\alpha, \beta \in \mathbb{R}$, and for any continuous functions f, g, h defined on $[a, b]$, we have the following.

(a) $f(x) + g(x) = g(x) + f(x)$

(b) $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$

(c) $\alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x)$

(d) $(\alpha + \beta)f(x) = \alpha f(x) + \beta f(x)$

(e) $(\alpha\beta)f(x) = \alpha(\beta f(x))$

(f) Let $\theta(x) = 0, \forall x \in [a, b]$. Then $f(x) + \theta(x) = f(x)$; $0f(x) = \theta(x)$;

(g) $1 \cdot f(x) = f(x)$

Definition above means that a vector space is closed under addition and scalar multiplication. We also call it *linear space* because it is closed under linear operations. To construct the notion of distance and convergence, we also define *metric*.

Definition. A metric space is a set S with a metric $\rho : S \times S \to \mathbb{R}$, such that for all $x, y, z \in S$:

(a) $\rho(x, y) \geq 0$ with equality iff $x = y$

(b) $\rho(x, y) = \rho(y, x)$

(c) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

Exercise 3.3.a. Show that (S, ρ) is a metric space for S being the set of integers with $\rho(x, y) = |x - y|$.
Solution: For any $x, y, z \in S$,

1. By definition of absolute value, $\rho(x, y) = |x - y| \geq 0$.
Equality holds when $\rho(x, x) = |x - x| = 0$ and $\rho(x, y) = 0 \Rightarrow |x - y| = 0 \Rightarrow x = y$.
2. $\rho(x, y) = |x - y| = |y - x| = \rho(y, x)$.

3. By the triangle inequality, $\rho(x,z) = |x-z| \leq |x-y| + |y-z| = \rho(x,y) + \rho(y,z)$.

We now focus on a specific form of metric that satisfies $\rho(x,y) = \rho(x-y,\theta)$. To do that, we define *norm*.

Definition. A normed vector space is a vector space S with a norm $||\cdot|| : S \to \mathbb{R}$ such that for all $x, y \in S$ and $\alpha \in \mathbb{R}$:

(a) $||x|| \geq 0$ with equality iff $x = \theta$

(b) $||\alpha x|| = |\alpha| ||x||$

(c) $||x+y|| \leq ||x|| + ||y||$ (also known as the triangular inequality)

Exercise 3.4. Show that the following are normed vector spaces.

(a) Let $S = \mathbb{R}^l$, with $||x|| = \left[\sum_{i=1}^l x_i^2\right]^{1/2}$.

(b) Let $S = \mathbb{R}^l$, with $||x|| = \max_i |x_i|$.

Solution: $\mathbb{R}^l$ is a vector space. The space of real numbers is closed under addition and scalar multiplication, and this holds for real vectors. We can designate $(0, \cdots, 0)$, a vector of l repeated zeros as the zero vector here. We focus on showing that the two norms in questions (a) and (b) satisfy the three conditions for norms. For all $x, y \in \mathbb{R}^l$ and $\alpha \in \mathbb{R}$, the following holds.

(a) 1. The sum of the squares of real numbers is nonnegative, so the square roots are nonnegative. Also, we have that $\|x\| = 0 \iff \left[\sum_{i=1}^l x_i^2\right]^{1/2} = 0 \iff \sum_{i=1}^l x_i^2 = 0 \iff x_i^2 = 0,\ \forall i$.

2. $\|\alpha x\| = \left[\sum_{i=1}^l (\alpha x_i)^2\right]^{1/2} = \left[\alpha^2 \sum_{i=1}^l x_i^2\right]^{1/2} = |\alpha| \left[\sum_{i=1}^l x_i^2\right]^{1/2} = |\alpha| \cdot \|x\|$

3. We use Cauchy-Schwarz inequality: $\left(\sum_{i=1}^l x_i y_i\right)^2 \leq \sum_{i=1}^l x_i^2 \sum_{i=1}^l y_i^2$ to show.

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\begin{aligned}
\|x+y\|^2 &= \sum_{i=1}^l (x_i + y_i)^2 \\
&= \sum_{i=1}^l \left(x_i^2 + 2x_i y_i + y_i^2\right) \\
&= \sum_{i=1}^l x_i^2 + 2\sum_{i=1}^l x_i y_i + \sum_{i=1}^l y_i^2 \\
&= \sum_{i=1}^l x_i^2 + 2\left[\left(\sum_{i=1}^l x_i y_i\right)^2\right]^{1/2} + \sum_{i=1}^l y_i^2 \\
&\leq \sum_{i=1}^l x_i^2 + 2\left[\sum_{i=1}^l x_i^2 \sum_{i=1}^l y_i^2\right]^{1/2} + \sum_{i=1}^l y_i^2 \\
&= \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2
\end{aligned}
$$

(b) 1. $\|x\| \geq 0$ because the maximum of nonnegative number (absolute values) is nonnegative. $\|x\| = 0 \iff \max_i |x_i| = 0 \iff |x_i| \leq 0, \forall i \iff x_i = 0, \forall i$.

2. $\|\alpha x\| = \max_i |\alpha x_i| = \max_i |\alpha \| x_i| = |\alpha | \max_i | x_i| = |\alpha | \cdot \| x\|$.
3. $\|x + y\| = \max_i |x_i + y_i| \leq \max_i (|x_i| + |y_i|) \leq \max_i |x_i| + \max_i |y_i| = \|x\| + \|y\|$

Any normed vector space $(S, ||\cdot||)$ is a metric space with $\rho(x, y) = ||x - y||, \forall x, y \in S$. Now we define the notion of convergence in a metric space.

Definition. A sequence $\{x_n\}_{n=0}^{\infty}$ in S converges to $x \in S$ if for each $\varepsilon > 0$, there exists N_ε such that $\rho(x_n, x) < \varepsilon, \forall n \geq N_\varepsilon$.

We need a candidate limit point x to verify convergence of a sequence, but luckily, we can also apply an alternative criterion if we don't have one.

Definition. A sequence $\{x_n\}_{n=0}^{\infty}$ in S is a Cauchy sequence if for each $\varepsilon > 0$, there exists N_ε such that $\rho(x_n, x_m) < \varepsilon, \forall n, m \geq N_\varepsilon$.

Exercise 3.5.b asks you to show that a converging sequence is a Cauchy sequence. It also holds, however, that a Cauchy sequence always converges in $\mathbb{R}^n$ equivalently. However, this may not be true depending on which metric space we are talking about. So we now discuss the space where a limit point of sequence exists.

Definition. A metric space (S, ρ) is complete if every Cauchy sequence in S converges to an element in S.

It follows that in complete metric spaces, verifying that a sequence is a Cauchy sequence is equivalent to verifying that a sequence converges to some point. For instance, $(\mathbb{R}^n, ||\cdot||)$ (Euclidean norm) is a complete metric space. A complete normed vector space is called a *Banach* space.

Theorem 3.1. Let $X \subseteq \mathbb{R}^l$ and let $C(X)$ be the set of bounded continuous functions $f : X \to \mathbb{R}$ with the sup norm, $||f|| = sup_{x\in X}|f(x)|$. Then $C(X)$ is a complete normed vector space.

Proof. $C(X)$ is a normed vector space. We now show that any Cauchy sequence $\{f_n\}$ in $C(X)$ converges to $f \in C(X)$. We find f first, show that $f_n \Rightarrow f$, and then show $f \in C(X)$.

For a fixed $x \in X$, following holds for the sequence $\{f_n(x)\}$:

$$|f_n(x) - f_m(x)| \leq \sup_{y\in X} |f_n(y) - f_m(y)| = ||f_n - f_m||$$

Since $\{f_n\}$ is a Cauchy sequence, the sequence of real numbers $\{f_n(x)\}$ is also a Cauchy sequence. By completeness of $\mathbb{R}$, this sequence has a limit point $f(x)$. For any value of $x \in X$, we can define a function $f : X \to \mathbb{R}$ accordingly.

Now for a given $\varepsilon > 0$, choose N_ε such that if $n, m \geq N_\varepsilon$, $||f_n - f_m|| \leq \varepsilon/2$. Then for any fixed $x \in X$ and $\forall m \geq n \geq N_\varepsilon$:

$$\begin{aligned}
|f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\
&\leq ||f_n - f_m|| + |f_m(x) - f(x)| \\
&\leq \varepsilon/2 + |f_m(x) - f(x)| \\
&\leq \varepsilon/2 + \varepsilon/2 = \varepsilon && \text{for a large enough } m \text{ because } f_m(x) \to f(x)
\end{aligned}$$

This holds for any $x \in X$, so $||f_n - f|| \leq \varepsilon, \forall n \geq N_\varepsilon$. That is, $\{f_n\}$ uniformly converges to f.

f is a bounded function because $\{f_n\}$ is bounded. Let $f_n(x) \leq B, \forall x \in X, \forall n$. Then for a given $\varepsilon > 0$ and any $x \in X$, there is some n_0 such that $|f_n(x) - f(x)| < \varepsilon$ for $\forall n \geq n_0$ because of uniform convergence. Then we have

$$|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| \leq \varepsilon + B$$

To show f is continuous, for a given $\varepsilon > 0$ and a given $x \in X$, choose k such that $||f - f_k|| < \varepsilon/3$. Then choose δ such that $||x - y||_E < \delta \Rightarrow |f_k(x) - f_k(y)| < \varepsilon/3$. Then we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\
&\leq 2||f - f_k|| + |f_k(x) - f_k(y)| \\
&\leq \varepsilon
\end{aligned}$$

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We have proved a set of bounded continuous functions with the sup norm to be a complete normed vector space, and this is the space in which we look for the value function in our dynamic programming. Section 3.2 proceeds to defining a contraction mapping T and shows that there is a unique fixed point of T in a complete metric space.