ECON 714A: Macroeconomic Theory - Handout 2

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Outline of SLP Chapter 3.2

- Throughout this chapter, we are trying to show the existence and uniqueness of solution v of dynamic programming problem (1). In the process, we construct an operator $T : C \to C$ defined as in (2). Finding solution v for (1) is equivalent to finding a fixed point of the mapping T in (2).

$$v(k) = \max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta v(y) \tag{1}$$

$$\Rightarrow Tv(k) = \max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta v(y) \tag{2}$$

- In Section 3.1, we have explored the properties of the function space C on which we will define T. In Section 3.2, we proceed to the sufficient conditions for T to satisfy to have a unique fixed point in C. We will show that T that satisfies Blackwell's sufficient conditions is a contraction in C, and has a unique fixed point in C by the Contraction Mapping Theorem.

SLP 3.2. The Contraction Mapping Theorem

Definition. Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping S into itself. T is a contraction mapping with modulus β if for some $\beta \in (0, 1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$ for all $x, y \in S$.

There is another way to verify if an operator is a contraction mapping other than directly using the definition.

Theorem 3.3. (Blackwell's Sufficient Conditions for a Contraction) Let $X \subseteq \mathbb{R}^l$ and let $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$ with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying

(a) *Monotonicity.* $f, g \in B(X)$ and $f(x) \leq g(x), \forall x \in X$ implies $(Tf)(x) \leq (Tg)(x), \forall x \in X$.

(b) *Discounting.* $\exists \beta \in (0, 1)$ such that $T(f + a)(x) \leq (Tf)(x) + \beta a, \forall f \in B(X), a \geq 0, x \in X$.

Then T is a contraction with modulus β.

Proof. For $\forall f, g \in B(X)$, $f \leq g + ||f - g||$. Then the following holds - the first inequality follows from (a) and the second follows from (b).

$$Tf \leq T(g + ||f - g||) \leq Tg + \beta||f - g||$$

We can apply this for $g \leq f + ||f - g||$ and obtain $Tg \leq Tf + \beta||f - g||$. Combining these two inequalities and applying the sup norm to $Tf - Tg$ and $Tg - Tf$, we get $||Tf - Tg|| \leq \beta||f - g||$. ■

Notation: If $f(x) \leq g(x), \forall x \in X$, we write $f \leq g$. Also, $(f + a)(x) = f(x) + a$.

We can verify if an operator T in a dynamic programming problem is a contraction using the Blackwell's sufficient conditions. Consider a following dynamic programming problem.

$$(T\nu)(k) = \max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta\nu(y)$$

(a) *Monotonicity:* For $\nu \leq \omega$, it is trivial that $T\nu \leq T\omega$.

(b) *Discounting:* See the following.

$$\begin{aligned} T(\nu + a)(k) &= \max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta(\nu(y) + a) \\ &= \max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta\nu(y) + \beta a \\ &= (T\nu)(k) + \beta a \end{aligned}$$

Hence, the operator T in this problem is a contraction of modulus β.

Another example of a contraction mapping is a continuous real function defined in $[a, b]$ that has a slope less than 1 in absolute value. Such function has a unique fixed point - we can verify this by drawing it on a Cartesian plane. We can generalize this result to the Contraction Mapping Theorem.

Theorem 3.2. (Contraction Mapping Theorem) If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then

(a) T has exactly one fixed point $\nu \in S$.

(b) For any $\nu_0 \in S$, $\rho(T^n\nu_0, \nu) \leq \beta^n\rho(\nu_0, \nu), \forall n \in \mathbb{N} \cup \{0\}$.

Proof. To prove (a), we first find a candidate ν, show that $T\nu = \nu$, and show its uniqueness.

For a contraction mapping T, define T^n by $T^0x = x, T(T^{n-1})x = T^nx, \forall n \geq 1$. Choose $\nu_0 \in S$ and define $\{\nu_n\}$ such that $\nu_n = T^n\nu_0, \nu_{n+1} = T\nu_n$. Since T is a contraction, $\rho(\nu_2, \nu_1) = \rho(T\nu_1, T\nu_0) \leq \beta\rho(\nu_1, \nu_0)$. By induction, we have $\rho(\nu_{n+1}, \nu_n) \leq \beta^n\rho(\nu_1, \nu_0)$. Now for any $m > n$, by triangular inequality, we have the following.

$$\begin{aligned} \rho(\nu_m, \nu_n) &\leq \rho(\nu_m, \nu_{m-1}) + \cdots + \rho(\nu_{n+1}, \nu_n) \\ &\leq \left(\beta^{m-1} + \cdots + \beta^n\right)\rho(\nu_1, \nu_0) \\ &\leq \frac{\beta^n}{1 - \beta}\rho(\nu_1, \nu_0) \end{aligned}$$

It is immediate that $\{\nu_n\}$ is a Cauchy sequence, and $\nu_n \to \nu$ for some $\nu \in S$ because S is complete. This is our candidate ν.

Now we show $T\nu = \nu$. For $\forall n, \forall\nu_0 \in S$, the following holds because $\nu_n \to \nu$.

$$\begin{aligned} \rho(T\nu, \nu) &\leq \rho(T\nu, T^n\nu_0) + \rho(T^n\nu_0, \nu) \\ &\leq \beta\rho(\nu, T^{n-1}\nu_0) + \rho(T^n\nu_0, \nu) \to 0 \ (n \to \infty) \end{aligned}$$

To show ν is uniquely defined, we prove that there is no other fixed point $\hat{\nu} \in S$ of T by contradiction. Suppose there is another fixed point $\hat{\nu} \neq \nu$.

$$0 < \rho(\nu, \hat{\nu}) = \rho(T\nu, T\hat{\nu}) \leq \beta\rho(\nu, \hat{\nu})$$

The inequality above cannot hold because $\beta < 1$. So ν is unique.

Now observe that for $\forall n \geq 1$, $\rho(T^n\nu_0, \nu) = \rho(T(T^{n-1})\nu_0, T\nu) \leq \beta\rho(T^{n-1}\nu_0, \nu)$. Apply induction to this inequality, and we obtain (b). ■

Note that if (S, ρ) is a complete metric space, (S', ρ) is a complete metric space for a closed subset S' of S. If $T : S \to S$ is a contraction mapping, and if it maps S' into S', then the unique fixed point of T on S is in S'. One usual approach is that you show the existence and uniqueness of fixed point in the bigger space S, and proceed to characterize it in S'.

Corollary 1. Let (S, ρ) be a complete metric space, and let $T : S \to S$ be a contraction mapping with fixed point $\nu \in S$. If S' is a closed subset of S and $T(S') \subseteq S'$, then $\nu \in S'$. Further, if $T(S') \subseteq S'' \subseteq S'$, then $\nu \in S''$.

Proof. Now pick any $\nu_0 \in S'$, and then $\{T^n\nu_0\}$ in S' is a Cauchy sequence in S' that converges to ν. Since S' is closed, $\nu \in S'$. This also proves completeness of (S', ρ). In addition, if $T(S') \subseteq S''$, then $\nu = T\nu \in S''$. ■

Part (a) of the Contraction Mapping Theorem guarantees that there is a unique solution of the dynamic programming problem we solve. We may not know what that solution is, but Part (b) offers a bound for the distance between $\{\nu_n\}$ that we construct and ν. But still, we do not know how large that bound would be without knowing what ν is. So we have a more realistic bound for $\rho(T^n\nu_0, \nu)$ as follows.

Exercise 3.9. Let (S, ρ), T, and ν be as given above, let β be the modulus of T, and let $\nu_0 \in S$. Show that

$$\rho(T^n\nu_0, \nu) \leq \frac{1}{1-\beta}\rho(T^n\nu_0, T^{n+1}\nu_0)$$

Solution: We use triangular inequality for proof.

$$\begin{aligned}
\rho(T^n\nu_0, \nu) &\leq \rho(T^n\nu_0, T^{n+1}\nu_0) + \rho(T^{n+1}\nu_0, \nu) \\
&= \rho(T^n\nu_0, T^{n+1}\nu_0) + \rho(T^{n+1}\nu_0, T\nu) \\
&\leq \rho(T^n\nu_0, T^{n+1}\nu_0) + \beta\rho(T^n\nu_0, \nu) \\
\therefore \rho(T^n\nu_0, \nu) &\leq \frac{1}{1-\beta}\rho(T^n\nu_0, T^{n+1}\nu_0)
\end{aligned}$$

Following is another application of the Contraction Mapping Theorem.

Corollary 2. Let (S, ρ) be a complete metric space, and let $T : S \to S$, and suppose that for some integer N, let $T^N : S \to S$ be a contraction mapping with modulus β. Then

(a) T has exactly one fixed point in S.

(b) For any $\nu_0 \in S$, $\rho(T^{kN}\nu_0, \nu) \leq \beta^k \rho(\nu_0, \nu), \forall k \geq 1.$

Based on what has been discussed until now, we further proceed to apply the Contraction Mapping Theorem in more generalized dynamic programming problems in Section 3.3. Our attention has been on the value function that we want to maximize until now, and we proceed to characterize the maximizing argument, or policy function, of dynamic programming problem.