ECON 714A: Macroeconomic Theory - Handout 3

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Outline of SLP Chapter 3.3 and SLP Chapter 4.1

- We want to ensure that the value function v and Tv defined below to be bounded and continuous on the set of state variables X. In Section 3.3, we characterize the restrictions on the feasibility set Γ and return function F that ensures such property, and that is called the Theorem of Maximum. To do that, we re-write the dynamic programming problem as below.

$$Tv(x) = \max_{y} \; F(x,y) + \beta v(y) \quad \text{s.t. } y \text{ is feasible given } x$$

$$\Rightarrow h(x) = \max_{y \in \Gamma(x)} \; f(x,y) \tag{1}$$

$$G(x) = \{y \in \Gamma(x) : f(x,y) = h(x)\} \tag{2}$$

  For each $x \in X$, if $f(x,\cdot)$ is continuous in y and the set $\Gamma(x)$ is nonempty and compact, there is a maximum of $f(x,\cdot)$ for each x. The Theorem of Maximum further proceeds to characterize conditions under which h and G are continuous in x.

- SLP Chapter 4.1 goes on to provide some conditions under which the solution of sequential maximization problem (SP) are equivalent to the solution of functional equation problem (FE).

$$\text{SP: } \max_{\{x_{t+1}\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$
$$\text{s.t. } \; x_{t+1} \in \Gamma(x_t), \forall t \geq 0, \; x_0 \text{ given.}$$

$$\text{FE: } v(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta v(y), \; \forall x \in X$$

  A general logic of proving the equivalence is showing that a sequence $\{x_{t+1}\}$ given x_0 solves the sequential problem if and only if it satisfies

$$v(x_t) = F(x_t, x_{t+1}) + \beta v(x_{t+1})$$

  This is shown in Theorems 4.2-5. Theorems 4.2 and 4.3 establish that the solution functions of (SP) and (FE) are equivalent when v is bounded, and Theorems 4.4 and 4.5 show that the policy functions are equivalent in the two problems, especially when they are bounded.

SLP 3.3. The Theorem of the Maximum

The feasible set Γ now maps an element in the state space x to a set, so it is a correspondence. We now define the notion of continuity for correspondences.

Definition. A correspondence $\Gamma : X \to Y$ is lower hemi-continuous (l.h.c.) at x if $\Gamma(x)$ is nonempty and if, for every $y \in \Gamma(x)$ and every sequence $x_n \to x$, there exists $N \geq 1$ and a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \to y$ and $y_n \in \Gamma(x_n)$ for $\forall n \geq N$.

Definition. A compact-valued correspondence $\Gamma : X \to Y$ is upper hemi-continuous (u.h.c.) at x if $\Gamma(x)$ is nonempty and if for every sequence $x_n \to x$ and every sequence $\{y_n\}$ such that $y_n \in \Gamma(x_n)$, all n, there exists a convergent subsequence of $\{y_n\}$ whose limit point y is in $\Gamma(x)$.

Definition. A correspondence $\Gamma : X \to Y$ is continuous at $x \in X$ if it is both u.h.c. and l.h.c. at x.

A correspondence $\Gamma : X \to Y$ is called l.h.c., u.h.c., or continuous if it satisfies the properties above at all $x \in X$. We can also define *graph* of Γ as the set $A = \{(x, y) \in X \times Y : y \in \Gamma(x)\}$. The following theorems provide sufficient conditions of A to ensure the upper and lower hemi-continuity of Γ.

Theorem. Let $\Gamma : X \to Y$ be a nonempty-valued correspondence, and let A be the graph of Γ. Suppose that A is closed, and that for any bounded set $\hat{X} \subseteq X$, the set $\Gamma(\hat{X})$ is bounded. Then Γ is compact-valued and upper hemi-continuous.

Theorem. Let $\Gamma : X \to Y$ be a nonempty-valued correspondence, and let A be the graph of Γ. Suppose that A is convex, and that for any bounded set $\hat{X} \subseteq X$, there is a bounded set $\hat{Y} \subseteq Y$ such that $\Gamma(x) \cap \hat{Y} \neq \varnothing, \forall x \in \hat{X}$. Then Γ is lower hemi-continuous at every interior point of X.

Using these theorems, we can now characterize the conditions for the value function h in (1) and the set of policy functions G in (2) to vary continuously on X.

Theorem. (Theorem of Maximum) Let $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^m$, let $f : X \times Y \to \mathbb{R}$ be a continuous function, and let $\Gamma : X \to Y$ be a compact-valued and continuous correspondence. Then the function $h : X \to \mathbb{R}$ defined in (1) and the correspondence $G : X \to Y$ defined in (2) is nonempty, compact-valued, and upper hemi-continuous.

Proof. First, we show that G is nonempty and compact-valued for each $x \in X$. Fix $x \in X$. Since the feasible set $\Gamma(x)$ is nonempty and compact, and $f(x, \cdot)$ is continuous, there is a maximum for (1) and the set $G(x)$ is nonempty. Since $G(x) \subseteq \Gamma(x)$ and $\Gamma(x)$ is compact, $G(x)$ is a bounded set. Now suppose $\{y_n\} \to y, \forall y_n \in G(x)$. Since $\Gamma(x)$ is closed, $y \in \Gamma(x)$. Since $h(x) = f(x, y_n), \forall n$ is a continuous function, $h(x) = f(x, y)$ and $y \in G(x)$. Hence, $G(x)$ is closed, so $G(x)$ is compact and nonempty for each x.

Now we show that $G(x)$ is upper hemi-continuous. Fix x and let $\{x_n\} \to x$. Choose $y_n \in G(x_n), \forall n$. Since Γ is upper hemi-continuous, $\exists \{y_{n_k}\} \to y \in \Gamma(x)$. Now let $z \in \Gamma(x)$. Then since Γ is lower hemi-continuous, $\exists \{z_{n_k}\} \to z$ with $z_{n_k} \in \Gamma(x_{n_k})$. We picked y_{n_k} from $G(x_{n_k})$, so $f(x_{n_k}, y_{n_k}) \geq f(x_{n_k}, z_{n_k})$ and $f(x, y) \geq f(x, z)$ by continuity of f. This holds for $\forall z \in \Gamma(x)$, it follows that $y \in G(x)$, so $G(x)$ is upper hemi-continuous.

Lastly, we show that h is continuous. Fix x and choose $\{x_n\} \to x$. Choose $y_n \in G(x_n)$ and let $\bar{h} = \limsup h(x_n)$ and $\underline{h} = \liminf h(x_n)$. Then there exists a subsequence $\{x_{n_k}\}$ such that $\bar{h} = \lim f(x_{n_k}, y_{n_k})$. Now, because G is upper hemi-continuous, we can pick a subsequence $\{y_j^{'}\}$ of $\{y_{n_k}\}$ that converges to $y \in G(x)$. Hence, $\bar{h} = \lim f(x_j, y_j^{'}) = f(x, y) = h(x)$. Analogously, we can show that $\underline{h} = h(x)$. Hence, $\{h(x_n)\} \to h(x)$. ■

From the Theorem of Maximum, we attain that the value function and the set of maximizing arguments of the dynamic programming problem vary continuously on the feasible set. Now assume that the feasible set Γ is convex and f is strictly concave in y. Then G is single-vlaued and is a continuous function, and we call this policy function. Lemma 3.7 of textbook establishes that if $f(x, y), y \in \Gamma(x)$ is close enough to $f(x, G(x))$, then this y is close to $G(x)$.

Lemma. Let $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^m$. Assume that the correspondence $\Gamma : X \to Y$ is nonempty, compact and convex-valued, and continuous, and let A be the graph of Γ. Assume that the function $f : A \to \mathbb{R}$ is continuous and that $f(x, \cdot)$ is strictly concave for each $x \in X$. Define the function $g : X \to Y$ by $g(x) = \arg\max_{y \in \Gamma(x)} f(x, y)$. Then for each $\varepsilon > 0$ and $x \in X$, there exists $\delta_x > 0$ such that

$$y \in \Gamma(x), |f(x, g(x)) - f(x, y)| < \delta_x \Rightarrow ||g(x) - y|| < \varepsilon$$

Theorem 3.8 of textbook proceeds that if $\{f_n\}$ is a sequence of continuous functions, each strictly concave in y, converging uniformly to f, then the policy functions $\{g_n\}$ converges pointwise to g. This means that once we have a set of guesses of value functions that (uniformly) converges to f, the associated set of policy functions $\{g_n\}$ will converge to g.

Theorem. Let X, Y, Γ, and A be as defined in Lemma 3.7. Let $\{f_n\}$ be a sequence of continuous real functions on A and assume that for each n and each $x \in X$, $f_n(x, \cdot)$ is strictly concave in its second argument. Assume that f has the same properties and that $f_n \to f$ uniformly. Define the functions g_n and g by

$$g_n(x) = \arg\max_{y \in \Gamma(x)} f_n(x, y)$$

$$g(x) = \arg\max_{y \in \Gamma(x)} f(x, y)$$

then $g_n \to g$ pointwise. If X is compact, $g_n \to g$ uniformly.

SLP 4.1. Principle of Optimality

We begin with two assumptions for the feasible set Γ and the set of plans $\Pi(x_0)$ given an initial state x_0. A plan $\tilde{x} = (x_0, x_1, \cdots)$ denotes a whole sequence satisfying the feasibility constraint in (SP).

Assumption 1. $\Gamma(x)$ is nonempty for all $x \in X$.

Assumption 2. For all $x_0 \in X$ and $\tilde{x} \in \Pi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists.

One preliminary result is that if X, Γ, F, and β satisfy Assumption 2, then we can decompose the infinite sum $u(\tilde{x}) = \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) = F(x_0, x_1) + \beta u(x')$ where $x' = (x_1, x_2, \cdots)$. Starting from the Theorem 4.2 below, we will define v^* as the supremum of $u(\tilde{x})$ that solves the sequential problem (SP). Theorem 4.2 yields that v^* that solves (SP) also solves (FE).

Theorem 4.2. Let X, Γ, F, and β satisfy Assumptions 1 and 2. Then the function $v^*(x_0) = \sup_{\tilde{x}\in\Pi(x_0)} u(\tilde{x})$ satisfies (FE).

Proof. If $\beta = 0$, it is trivial. Suppose $\beta > 0$ and pick $x_0 \in X$. First, suppose $v^*(x_0)$ is finite. Then we have

$$\begin{aligned}
&v^*(x_0) \geq u(\tilde{x}), \forall \tilde{x} \in \Pi(x_0)\\
&v^*(x_0) \leq u(\tilde{x}) + \epsilon, \forall \epsilon > 0 \text{ and for some } \tilde{x} \in \Pi(x_0)\\
&\exists\, x' = (x_1, x_2, \cdots) \text{ s.t. } u(x') \geq v^*(x_1) - \epsilon\\
\Rightarrow\ &v^*(x_0) \geq F(x_0, x_1) + \beta u(x') \geq F(x_0, x_1) + \beta v^*(x_1) - \beta\epsilon\\
\Rightarrow\ &v^*(x_0) \geq F(x_0, y) + \beta v^*(y), \forall y \in \Gamma(x_0)
\end{aligned}$$

Now to show that v^* is the least upper bound defined as $\sup_{y\in\Gamma(x)} F(x, y) + \beta v(y)$, we can choose a plan $\tilde{x} \in \Pi(x_0)$ and show the following:

$$\begin{aligned}
v^*(x_0) &\leq u(\tilde{x}) + \epsilon = F(x_0, x_1) + \beta u(x') + \epsilon\\
&\leq F(x_0, x_1) + \beta v^*(x_1) + \epsilon,\ x_1 \in \Gamma(x_0)
\end{aligned}$$

Second, consider the case of $v^*(x) = \infty$. Then there exists a sequence $\{\tilde{x}^k\} \in \Pi(x_0)$ such that $\lim_{k\to\infty} u(\tilde{x}^k) = \infty$. Then since $x_1^k \in \Gamma(x_0), \forall k$, we have the following.

$$u(\tilde{x}^k) = F(x_0, x_1^k) + \beta u(\tilde{x}^{'k}) \leq F(x_0, x_1^k) + \beta v^*(x_1^k), \forall k$$

The right hand side will tend to infinity.

Lastly, consider the case of $v^*(x) = -\infty$. Then we have that $v^*(x_0) \geq u(\tilde{x}) = F(x_0, x_1) + \beta u(x') = -\infty$ for $\forall \tilde{x} \in \Pi(x_0)$. Since F is real-valued, this means that $u(x') = -\infty$, for $\forall x_1 \in \Gamma(x_0), \forall x' \in \Pi(x_1)$. Hence, $v^*(x_1) = -\infty, \forall x_1 \in \Gamma(x_0)$. This yields $F(x_0, y) + v^*(y) = -\infty, \forall y \in \Gamma(x_0)$. ■

Theorem 4.3 yields a partial converse to Theorem 4.2 in that a solution to (FE) that also satisfies a boundedness condition is a solution to (SP).

Theorem 4.3. Let X, Γ, F, and β satisfy Assumptions 1 and 2. If v is a solution to (FE) and satisfies $\lim_{n\to\infty} \beta^n v(x_n) = 0, \forall (x_0, x_1, \cdots) \in \Pi(x_0), \forall x_0 \in X$, then $v = v^*$.

Proof. First consider the case $v(x_0)$ is finite. Then by construction, for $\forall \tilde{x} \in \Pi(x_0)$,

$$\begin{aligned}
v(x_0) &\geq F(x_0, x_1) + \beta v(x_1)\\
&\geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 v(x_2)\\
\cdots &\geq \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} v(x_{n+1}) \to u(\tilde{x})
\end{aligned}$$

Also, fix $\epsilon > 0$ and choose $\{\delta_t\}_{t=1}^{\infty} \in \mathbb{R}_+$ such that $\sum_{t=1}^{\infty} \beta^{t-1}\delta_t \leq \epsilon/2$. Then we can choose $x_1 \in \Gamma(x_0)$, $x_2 \in \Gamma(x_1), \cdots$ such that

$$\begin{aligned} v(x_t) &\leq F(x_t, x_{t+1}) + \beta v(x_{t+1}) + \delta_{t+1}, \forall t \geq 0 \\ \Rightarrow v(x_0) &\leq \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} v(x_{n+1}) + \sum_{t=0}^{n} \beta^t \delta_{t+1} \\ &\leq u(\tilde{x}) + \epsilon/2 \end{aligned}$$

Hence, we have that v solves (SP) as well when it is finite. Also note that the boundedness condition rules out the case of v being $-\infty$. Lastly, consider the case when $v(x_0) = \infty$. Choose $n \geq 0$ and pick a plan $(x_0, x_1, \cdots, x_n)$ such that $x_t \in \Gamma(x_{t-1}), v(x_t) = \infty$ for $t = 0, 1, \cdots, n$ and $v(x_{n+1}) < \infty$ for $\forall x_{n+1} \in \Gamma(x_n)$. Note that by boundedness condition, $n < \infty$. Now fix $A > 0$ and choose $x_{n+1}^A \in \Gamma(x_n)$ such that

$$F(x_n, x_{n+1}^A) + \beta v(x_{n+1}^A) \geq \beta^{-n} \left[A + 1 - \sum_{t=0}^{n-1} \beta^t F(x_t, x_{t+1}) \right]$$

Now choose a plan $\tilde{x}_{n+1}^A$ starting from x_{n+1}^A such that $v(x_{n+1}^A) \leq u(\tilde{x}_{n+1}^A) + \beta^{-(n+1)}$. Then we have for $\tilde{x}^A = (x_0, x_1, \cdots, x_n, \tilde{x}_{n+1}^A) \in \Pi(x_0)$,

$$\begin{aligned} u(\tilde{x}^A) &= \sum_{t=0}^{n-1} \beta^t F(x_t, x_{t+1}) + \beta^n F(x_n, x_{n+1}^A) + \beta^{n+1} u(\tilde{x}_{n+1}^A) \\ &= \sum_{t=0}^{n-1} \beta^t F(x_t, x_{t+1}) + \beta^n F(x_n, x_{n+1}^A) + \beta^{n+1} v(x_{n+1}^A) + \beta^{n+1} u(\tilde{x}_{n+1}^A) - \beta^{n+1} v(x_{n+1}^A) \\ &\geq A \end{aligned}$$

$A > 0$ was an arbitrary number, so $v^*(x_0) = \infty = v(x_0)$. ■

Now we have that the solutions to (SP) and (FE) are equivalent, we proceed to show that the optimal plans in (SP) and (FE) are equivalent for $v^* = v$. Theorem 4.4 shows that a solution to (SP) is a solution to (FE).

Theorem 4.4. Let X, Γ, F, and β satisfy Assumptions 1 and 2. Let $\tilde{x}^* \in \Pi(x_0)$ be a feasible plan that attains the supremum in (SP) for an initial state x_0. Then $v^*(x_t) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*)$ holds.

Proof. Since x^* attains the supremum in (SP), we have

$$\begin{aligned} v^*(x_0^*) = u(x^*) &= F(x_0, x_1^*) + \beta u(\tilde{x}_1^*) \\ &\geq F(x_0, x_1^*) + \beta u(\tilde{x}_1^{'}), \forall \tilde{x}_1^{'} = (x_1^*, x_2, x_3, \cdots) \in \Pi(x_1^*) \end{aligned}$$

Note that $\tilde{x}_1^* = (x_1^*, x_2^*, \cdots)$. The inequality holds because $(x_0, x_1^*, x_2, x_3, \cdots) \in \Pi(x_0)$, and we have $u(\tilde{x}_1^*) \geq u(\tilde{x}_1^{'}), \forall \tilde{x}_1^{'}$ and hence, $u(\tilde{x}_1^*) = v^*(x_1^*)$. This yields the (FE) condition to hold when $t = 0$. Apply mathematical induction on this and we can generalize the result to $\forall t$. ■

Finally, Theorem 4.5 offers a partial converse to Theorem 4.4 that a solution of (FE) that satisfies a boundedness condition solves (SP).

Theorem 4.5. Let X, Γ, F, and β satisfy Assumptions 1 and 2. Let $\tilde{x}^* \in \Pi(x_0)$ be a feasible plan from x_0 satisfying $v^*(x_t) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*)$ and $\limsup_{t \to \infty} \beta^t v^*(x_t^*) \leq 0$. Then x^* attains the supremum in (SP) for initial state x_0.

Proof. Suppose that $\tilde{x}^* \in \Pi(x_0)$ satisfies the two conditions. Then by induction on $v^*(x_t) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*)$, we have that

$$v^*(x_0) = \sum_{t=0}^{n} \beta^t F(x_t^*, x_{t+1}^*) + \beta^{n+1} v^*(x_{t+1}^*), \forall n \geq 1$$

Taking the limits, we find that $v^*(x_0) \leq u(\tilde{x}^*)$. Since $\tilde{x}^* \in \Pi(x_0)$, $v^*(x_0) \geq u(\tilde{x}^*)$ by construction of v^*. Hence, x^* attains the supremum in (SP). ■