ECON 714A: Macroeconomic Theory - Handout 4

Professor Job Boerma

TA: Yeonggyu Yun

February 9th, 2023

Recap of SLP Chapter 4.1

Theorems 4.2 and 4.3 establish that the solution functions of (SP) and (FE) are equivalent when v is bounded, and Theorems 4.4 and 4.5 show that the policy functions are equivalent in the two problems, especially when they are bounded.

$$\text{SP: } \max_{\{x_{t+1}\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$
$$\text{s.t. } x_{t+1} \in \Gamma(x_t), \forall t \geq 0, \; x_0 \text{ given.}$$

$$\text{FE: } v(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y), \; \forall x \in X$$

In a practical sense, focus on proving Theorems 4.2 and 4.3 when v is finite-valued. The proof for the finite case consists two parts: (1) you show that v is an upper bound of either (SP) or (FE) in the two theorems, respectively, and (2) then show that there exists some u such that $v \leq u + \epsilon$ for a small positive ϵ so that v is the least upper bound.

Provided that (SP) are (FE) are equivalent, Chapter 4.1, together with Chapter 3, offers a background on which we can solve the sequential utility/profit maximization of a private agent in economic models by solving a functional equation problem via value function iteration. Existence and uniqueness of the value function is provided by the Contraction Mapping Theorem (and Blackwell's sufficient conditions). Then the Theorem of Maximum provides that the sequence of value functions that we will iterate will be continuous, and that it will converge to the solution we are looking for.

Consumption-Savings Choice Model from LS Chapter 17 [1]

We consider a consumer maximizing his lifetime utility in an incomplete markets economy. The consumer is subject to borrowing constraints and he can purchase only a state non-contingent claim. We study a deterministic version of the model.

A representative consumer maximizes his expected lifetime utility $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ where $\beta \in (0, 1)$ and $u(c)$ is a strictly increasing, strictly concave, and twice continuously differentiable. It also satisfies the Inada condition $\lim_{c \to 0} u'(c) = \infty$. The agent gets an infinite sequence of endowment

[1] Please refer to Sections 17.1-3 of Ljungqvist and Sargent textbook.

$\{y_t\}_{t=0}^{\infty}$ where each endowment y_t takes one of a finite series $\{\bar{y}_1, \cdots, \bar{y}_S\}$. The agent chooses consumption c_t and saving a_{t+1} each period where the rate of return for this asset is given as r that satisfies $\beta(1+r) = 1$. Hence, the sequential problem can be written as below.

$$\begin{aligned} &\max_{c_t, a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\text{s.t. } a_{t+1} = (1+r)(a_t - c_t) + y_{t+1} \\ &\quad 0 \le c_t \le a_t, \ a_0 \text{ given.} \end{aligned}$$

We can write this problem as a recursive problem with the Bellman equation as below. Note that $V(a)$ is increasing, strictly concave, and differentiable in a.

$$V(a) = \max_{0 \le c \le a} u(c) + \beta V\left((1+r)(a-c) + y\right)$$

We are confining ourselves to a deterministic environment where the endowment process is non-stochastic. Hence there is no perpetual growth in income, so the consumption choice c converges to some value. However, even though there is no randomness, the agent is subject to the borrowing constraint so we have to consider two separate cases depending on whether or not the constraint binds.

Case 1: Natural Debt Limit

We now denote b_t as the amount of one-period debt of the consumer: $b_t = y_t - a_t$ with $b_0 = 0$. Using this notation, the sequential budget constraint changes to $c_t + b_t \le \beta b_{t+1} + y_t$. Natural borrowing limit can be obtained by combining the budget constraints starting from period t while setting $c_t = 0, \forall t$. That implies that the natural borrowing limit is the maximum amount of debt that the consumer can borrow and repay: it is the level of debt he can pay back by consuming nothing.

$$\begin{aligned} b_t &\le \beta b_{t+1} + y_t \\ \beta b_{t+1} &\le \beta^2 b_{t+2} + \beta y_{t+1} \\ (\cdots) \Rightarrow \quad b_t &\le \sum_{j=0}^{\infty} \beta^j y_{t+j} \equiv \bar{b}_t \end{aligned}$$

Now combining all the sequential budget constraints with $b_0 = 0$ and imposing the transversality condition $\lim_{T\to\infty} \beta^{T+1} b_{T+1} = 0$, we get the following constraint for the sequence of consumption c_t. This is the only restriction that the consumption sequence should satisfy under the natural borrowing constraints because we have characterized the natural borrowing limit in each period exactly by summing up all the budget constraints starting from each period.

$$\begin{aligned} \sum_{t=0}^{T} \beta^t c_t + \beta^{T+1} b_{T+1} &\le \sum_{t=0}^{T} \beta^t y_t \\ \sum_{t=0}^{\infty} \beta^t c_t &\le \sum_{t=0}^{\infty} \beta^t y_t \end{aligned}$$

Hence, we can solve for the Euler equation in the sequential problem subject to the time-0 constraint on consumption above.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left(\sum_{t=0}^{\infty} \beta^t y_t - \sum_{t=0}^{\infty} \beta^t c_t \right) + \sum_{t=0}^{\infty} \left[\gamma_t(\bar{b}_t - b_t) + \delta_t(\beta b_{t+1} + y_t - b_t - c_t) \right]$$

$$\text{F.O.C.} \Rightarrow \beta^t u'(c_t) = \beta^t \lambda + \delta_t$$

$$\beta \delta_t - \delta_{t+1} - \gamma_{t+1} = 0$$

$$\Rightarrow \beta^{t+1}(u'(c_t) - \lambda) = \beta^{t+1}(u'(c_{t+1}) - \lambda) + \gamma_{t+1}$$

We have that $u'(c_t) = u'(c_{t+1})$ if the natural borrowing constraint does not bind. In other words, we can attain $c_t = \bar{c}, \forall t$ if the borrowing limit is not violated. From this, we can characterize the consumption and savings choices in every period as follows. (I am not saying that this is a unique solution.) Assume that the borrowing constraint never binds, then the optimal consumption will be that $c_t = \bar{c}, \forall t$. Then we plug in the value of $\bar{c}$ to the budget constraint in period-t value starting from period t to the future.

$$\sum_{t=0}^{\infty} \beta^t \bar{c} = \frac{\bar{c}}{1-\beta} = \sum_{t=0}^{\infty} \beta^t y_t$$

$$\text{and } \sum_{j=0}^{\infty} \beta^j c_{t+j} + b_t = \sum_{j=0}^{\infty} \beta^j y_{t+j}$$

$$b_t = \sum_{j=0}^{\infty} \beta^j (y_{t+j} - \bar{c})$$

$$= \sum_{j=0}^{\infty} \beta^j y_{t+j} - \sum_{j=0}^{\infty} \beta^j y_j < \sum_{j=0}^{\infty} \beta^j y_{t+j} = \bar{b}_t$$

The last equality ensures that the natural borrowing constraint is satisfied for all periods. Note that b_t can be either positive or negative depending on the realization of endowments.

Case 2: Borrowing Constraint

Now we consider that the consumer is subject to the borrowing constraint $\beta b_{t+1} \leq 0 . \forall t$, so the consumer cannot borrow. Also consider the case that this borrowing constraint binds for finite periods at most. Note that the first order condition of the sequential problem will be obtained in a similar way as before: $u'(c_t) \geq u'(c_{t+1})$. The equality holds only when $b_{t+1} < 0$. Hence, we have that either $c_t = c_{t-1}$ or $c_t > c_{t-1}$ with $b_t = 0, a_t = y_t$. The latter case is when the borrowing constraint binds.

Combining the sequential budget constraints from time t, we have the following. The second equation holds if the borrowing constraint was binding at $t-1$.

$$\sum_{j=0}^{\infty} \beta^j c_{t+j} = a_t + \sum_{j=1}^{\infty} \beta^j y_{t+j}$$

$$\text{If } b_t = 0, \sum_{j=0}^{\infty} \beta^j c_{t+j} = \sum_{j=0}^{\infty} \beta^j y_{t+j}$$

If the consumer with zero savings at t knows that the borrowing constraint will never bind again, he will smooth the consumption at level $\tilde{c}$ such that $\frac{\tilde{c}}{1-\beta} = \sum_{j=0}^{\infty} \beta^j y_{t+j}$. Also, the consumption sequence is weakly increasing by the first-order condition with an increase happening only when the borrowing constraint binds. Hence, the consumption sequence will be a discrete time step function that jumps when the borrowing constraint binds.

We now show that consumption sequence will converge to the supremum of the annuity return on the present value of income starting from current period. Note that the annuity return on the present value of income $x(t)$ is derived as follows.

$$\begin{aligned}\sum_{j=t}^{\infty}\beta^{j-t}x(t) &= \sum_{j=t}^{\infty}\beta^{j-t}y_j \\ \frac{x(t)}{1-\beta} &= \sum_{j=t}^{\infty}\beta^{j-t}y_j \\ \left(\beta=\frac{1}{1+r}\right)\Rightarrow x(t) &= \frac{r}{1+r}\sum_{j=t}^{\infty}(1+r)^{t-j}y_j\end{aligned}$$

Intuitively, the consumption sequence will converge to $\tilde{c}$ if the binding constraint never binds after period $t-1$ and this $\tilde{c}$ is the annuity return on the time-t value of income. Now the following proposition says that the $\tilde{c}$ is the supremum of $x(t)$ over all t.

Proposition. Given a borrowing constraint and a nonstochastic endowment stream, the limit of the nondecreasing optimal consumption path converges, and

$$\tilde{c} = \lim_{t\to\infty} c_t = \sup_t x(t) = \bar{x}$$

Proof. Denote c_j below as the optimal consumption sequence. We first show that $\tilde{c}\leq\bar{x}$ and prove it by contradiction. Assume $\tilde{c}>\bar{x}$. Then from the first order condition, we have that there is a t such that the borrowing constraint binds at $t-1$ ($b_t=0$) and $x(t)\leq\bar{x}<c_j^*,\forall j\geq t$. This implies that $\sum_{j=t}^{\infty}\beta^{j-t}c_j>\sum_{j=t}^{\infty}\beta^{j-t}x(t)=\sum_{j=t}^{\infty}\beta^{j-t}y_j$, so there is a large enough τ such that

$$\begin{aligned}\sum_{j=t}^{\tau}\beta^{j-t}c_j &> \sum_{j=t}^{\tau}\beta^{j-t}y_j \\ \beta^{\tau-t+1}b_{\tau+1} = \sum_{j=t}^{\tau}\beta^{j-t}c_j &- \sum_{j=t}^{\tau}\beta^{j-t}y_j > 0\end{aligned}$$

This is contradictory to the borrowing constraint $b_t\leq 0,\forall t$. So $\tilde{c}\leq\bar{x}$.

We secondly show that $\tilde{c}\geq\bar{x}$ by contradiction. Assume $\tilde{c}<\bar{x}$. Then there is a $x(t)$ such that $c_j\leq x(t),\forall j\geq t$. This implies the following.

$$\begin{aligned}\sum_{j=t}^{\infty}\beta^{j-t}c_j < \sum_{j=t}^{\infty}\beta^{j-t}x(t) &= \sum_{j=t}^{\infty}\beta^{j-t}y_j \\ &\leq y_t - b_t + \sum_{j=t+1}^{\infty}\beta^{j-t}y_j\end{aligned}$$

The last inequality holds because $b_t\leq 0$. Hence, there exists a small $\epsilon>0$ and some $\hat{\tau}>t$ such that for $\forall\tau > \hat{\tau}$,

$$\sum_{j=t}^{\tau} \beta^{j-t} c_j < y_t - b_t + \sum_{j=t+1}^{\tau} \beta^{j-t} y_j - \epsilon$$
$$\beta^{\tau-t} c_\tau < \beta^{\tau-t}(y_\tau - b_\tau) - \epsilon$$

Add and subtract $\{\beta^k b_{t+k}\}_{k=1}^{\tau-t}$, and the last inequality is derived. So we have that for $\forall\tau > \hat{\tau}$, the consumption sequence satisfies b_τ to be an arbitrarily small negative value and be feasible. Then, we can construct an alternative sequence that adds even smaller arbitrary value to consumption, say $0.5\beta^{t-\tau}\epsilon$, for $\forall\tau > \hat{\tau}$ and attain higher utility. So $\{c_j\}$ is not optimal and this is a contradiction. Hence, $\tilde{c} \geq \bar{x}$. Combining the two results, we have $\tilde{c} = \bar{x}$. ■