ECON 714A: Macroeconomic Theory - Handout 5

Professor Job Boerma

TA: Yeonggyu Yun

February 23, 2023

1 McCall (1970) Job Search Model

We consider an unemployed worker searching for a job following McCall (1970).[1] The worker draws a wage offer $w \sim F(w)$ where $F(0) = 0$ and $F(B) = 1, B < \infty$. The worker can either accept the offer or reject the offer, but if he rejects the offer, he gets unemployment benefit of c this period and draws another wage offer next period. Once he is employed, there is no quitting or getting fired from the job and the worker earns w each period henceforth.

This is an infinite horizon model and the worker maximizes his lifetime utility $\sum_{t=0}^{\infty} \beta^t y_t$. We denote the expected value of $\sum_{t=0}^{\infty} \beta^t y_t$ as $v(w)$, which is the lifetime utility of a worker when we gets an offer of w in the market. The value function $v(w)$ hence satisfies the following Bellman equation.

$$v(w) = \max \left\{ \frac{w}{1-\beta},\ c + \beta \int_0^B v(w')dF(w') \right\}$$

Note that the value function satisfies reservation property. That is, there is a unique cut-off level of $\bar{w}$ where the two terms in the parenthesis are equal. We call that level of wage *reservation wage*. The reason why reservation property holds in this setting is that the left term is increasing in w while the right term is a constant. Hence, we can rewrite the value function as follows.

$$v(w) = \begin{cases} c + \beta \int_0^B v(w')dF(w') = \frac{\bar{w}}{1-\beta} & \text{if } w \leq \bar{w} \\ \frac{w}{1-\beta} & \text{if } w \geq \bar{w} \end{cases}$$

The reservation wage $\bar{w}$ satisfies the following.

$$\frac{\bar{w}}{1-\beta} = c + \beta \int_0^B v(w')dF(w')$$

We proceed to characterize the reservation wage $\bar{w}$ and discuss how it varies with different levels of c and mean-preserving change in variance of F.

[1]The notation and discussion follows Section 6.3 of Ljungqvist and Sargent (2018).

1.1 Characterizing Reservation Wage

Reservation wage $\bar{w}$ is characterized as the value that equalizes the two terms in the Bellman equation. Hence we can write it in this way.

$$\begin{aligned} \frac{\bar{w}}{1-\beta} &= c + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} dF(w') + \beta \int_{\bar{w}}^{B} \frac{w'}{1-\beta} dF(w') \\ \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} dF(w') + \int_{\bar{w}}^{B} \frac{\bar{w}}{1-\beta} dF(w') &= c + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} dF(w') + \beta \int_{\bar{w}}^{B} \frac{w'}{1-\beta} dF(w') \\ \Rightarrow \bar{w} - c &= \frac{\beta}{1-\beta} \int_{\bar{w}}^{B} (w' - \bar{w}) dF(w') \end{aligned}$$

Looking at the last equation, the left hand side is the cost of rejecting the offer $\bar{w}$ now and searching for another job next period. The right hand side is the gains from searching next period and getting a wage higher than $\bar{w}$.

Now define $h(w) = \frac{\beta}{1-\beta} \int_w^B (w' - w) dF(w')$. We have that $h(0) = \frac{\beta}{1-\beta} Ew$ and $h(B) = 0$. Applying Leibniz's rule (see page 163-164 of the textbook), we have the following.

$$\begin{aligned} h'(w) &= -\frac{\beta}{1-\beta} \int_w^B dF(w') = -\frac{\beta}{1-\beta}[1 - F(w)] < 0 \\ h''(w) &= \frac{\beta}{1-\beta} F'(w) > 0 \end{aligned}$$

We obtain that $h(w)$ is a decreasing, strictly convex function of w. Comparing this $h(w)$ function and $w - c$, we obtain that the reservation wage is increasing in c.

1.2 Mean-preserving Spreads

We can characterize the reservation wage in an alternative way.

$$\begin{aligned} \bar{w} - c &= \frac{\beta}{1-\beta} \int_{\bar{w}}^{B} (w' - \bar{w}) dF(w') + \frac{\beta}{1-\beta} \int_0^{\bar{w}} (w' - \bar{w}) dF(w') - \frac{\beta}{1-\beta} \int_0^{\bar{w}} (w' - \bar{w}) dF(w') \\ &= \frac{\beta}{1-\beta} E(w) - \frac{\beta}{1-\beta} \bar{w} - \frac{\beta}{1-\beta} \int_0^{\bar{w}} (w' - \bar{w}) dF(w') \\ \Rightarrow \bar{w} - (1-\beta)c &= \beta E(w) - \beta \int_0^{\bar{w}} (w' - \bar{w}) dF(w') \\ \Rightarrow \bar{w} - c &= \beta(E(w) - c) + \beta \int_0^{\bar{w}} F(w') dw' \end{aligned}$$

Define $g(s) = \int_0^s F(p) dp$, then this function is an increasing convex function, and the equation above can be written as $\bar{w} - c = \beta(Ew - c) + \beta g(\bar{w})$. Now consider a mean-preserving increase in risk in the wage distribution. That increases $g(w)$ for all levels of w.

Definition. The distribution F_1 is second-order stochastic dominant over F_2 if and only if $S_1(W) < S_2(W)$ for all $W \in (a, b)$ and $S_1(b) = S_2(b)$, where

$$S(W) = \int_a^W F(w) dw.$$

F_2 is going to be riskier than F_1, we also say that F_2 is a mean-preserving spread of F_1.

$\bar{w} - c = \beta(Ew - c) + \beta g(\bar{w})$ implies that the increase in mean-preserving spreads of F increases the reservation wage. Intuitively, this is because it increases the value of outside-option of the worker to reject to the offer because the worker now faces a higher incidence of wage offers higher than $\bar{w}$. Hence, the reservation wage should be higher to match this outside option value.

1.3 Waiting Time to Employment

Now denote N be the random variable for the length of time until the worker accepts the offer. $N = 1$ means that the first job offer is accepted. Also denote $\lambda = \int_0^{\bar{w}} dF(w')$, the probability that a job offer is rejected or the worker draws a wage lower than $\bar{w}$ in other words. Then N follows a geometric distribution with success probability $1 - \lambda$: $Pr(N = j) = \lambda^j(1 - \lambda)$. Then the mean waiting time can be obtained as follows.

$$\begin{aligned}
E(N) &= \sum_{j=1}^{\infty} j(1-\lambda)\lambda^j = (1-\lambda)\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\lambda^{k+j-1} \\
&= (1-\lambda)\sum_{k=0}^{\infty}\frac{\lambda^k}{1-\lambda} = \sum_{k=0}^{\infty}\lambda^k \\
&= \frac{1}{1-\lambda}
\end{aligned}$$

1.4 Firing

Now consider that the worker can be fired with probability $\alpha \in (0, 1)$ after getting employed. Denote $\hat{v}(w)$ as the expected present value of income of an unemployed worker with wage w. If he accepts the wage offer w, he earns w this period. The next period, he gets fired by probability α and gets unemployment benefit c and draw a job offer the next period (two periods from now). If he does not get fired by probability $1 - \alpha$, he gets the same expected value from the income stream $\hat{v}(w)$ next period because he is subject to the exactly same risk of getting fired or not afterwards. Hence, the value of accepting a job offer can be written as:

$$\hat{v}(w) = w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha\left[c + \beta\int_0^B \hat{v}(w')dF(w')\right]$$

If the worker rejects the offer now, he gets $c + \beta E\hat{v}$. Hence, the Bellman equation now changes as:

$$\begin{aligned}
\hat{v}(w) &= \max\left\{w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha\left[c + \beta E\hat{v}\right], c + \beta E\hat{v}\right\} \\
&= \begin{cases} c + \beta E\hat{v} & \text{if } w \leq \tilde{w} \\ \frac{w + \beta\alpha(c + \beta E\hat{v})}{1 - \beta(1-\alpha)} & \text{if } w \geq \tilde{w} \end{cases}
\end{aligned}$$

The same logic applies to prove the reservation property here, but one thing to be careful is that $\hat{v}(w)$ appears in the left term inside the parenthesis. But we can still show that $\hat{v}(w)$ is weakly increasing in w with proof by contradiction. Assume $\hat{v}'(w) < 0$ almost everywhere except for a countable number of points then it should be either $\beta > 1$ or $\hat{v}'(w) = 0$, which is false. I omit the formal proof here, but the sketch is as follows.

Assume $\hat{v}'(w) < 0$ almost everywhere except for a countable number of points (where $\hat{v}'$ does not exist), then it must be that $\hat{v}(0) > c + \beta E\hat{v}$. Otherwise, there $\hat{v}$ is a constant function with $\hat{v}' = 0$, which is a contradiction trivially. Then we have the following.

$$\begin{aligned}\hat{v}(0) = \beta(1-\alpha)\hat{v}(0) + \beta\alpha(c+\beta E\hat{v}) &> c + \beta E\hat{v} \\ \frac{\beta\alpha}{1-\beta(1-\alpha)}(c+\beta E\hat{v}) &> c+\beta E\hat{v} \\ \beta &> 1\end{aligned}$$

Hence, we reach a contraction and $\hat{v}(w)$ is a (weakly) increasing function in w.

We can characterize the reservation wage $\tilde{w}$ as following.

$$\begin{aligned}\frac{\tilde{w}+\beta\alpha(c+\beta E\hat{v})}{1-\beta(1-\alpha)} &= c+\beta E\hat{v} \\ \frac{\tilde{w}}{1-\beta} &= c+\beta\int_0^B \hat{v}(w')dF(w')\end{aligned}$$

We have a similar expression as the characterization of $\bar{w}$, but note that $\tilde{w} < \bar{w}$ because $\hat{v}(w) < v(w), \forall w$. So firing possibility reduces the reservation wage because the workers now expect each job to last for a shorter period of time, so the benefit of waiting for a better offer decreases.

Technically, $\hat{v}(w) < v(w)$ follows from an argument that the exact income stream with firing probability can be generated under a model where we allow workers to quit the job and search another one, but even then, the workers would not choose to quit the jobs once they accept it.

Consider an alternative setting where the worker with wage offer w can quit the job and search for another one and there is no firing probability. Then the payoff of accepting the job and not quitting is $\frac{w}{1-\beta}$. Payoff of rejecting the job is $\frac{\bar{w}}{1-\beta}$ as in the baseline model. If the worker accepts the job but quits after t periods, then his payoff can be written as:

$$\frac{w}{1-\beta} - \frac{\beta^t w}{1-\beta} + \beta^t\left(c+\beta\int_0^B v(w')dF(w')\right) = \frac{w}{1-\beta} - \beta^t\frac{w-\bar{w}}{1-\beta}$$

Hence with $w \geq \bar{w}$, quitting the job is always inferior to sticking to the job forever. The worker never chooses to quit the job once he accepts it.

Coming back to our discussion on firing probability, any income flow under firing risk can be mimicked in the model where workers can choose to quit the job. But, it is never optimal for the workers to quit the job voluntarily. This means the value of income under firing risks is always less than the value of income in the baseline model without firing and quitting. Hence we have $\hat{v}(w) < v(w)$.

1.5 Computation Outline

We can solve the model numerically by value function iteration. Please refer to the pseudo-algorithm below.[2]

Algorithm 1: McCall Job Search Model

Set up parameters β, c, and F, and a grid of state space of $w \in [0, B]$
(Discretize the state space W and distribution F)
Assign value function $v(w)$ and policy function $e(w) \in \{\text{Accept, Reject}\}$ for each w
Make a initial guess of value functions $v(w)$
while *error* $>$ *tol* **do**
    foreach w **do**
        $v_{new}(w) = \max\left\{\frac{w}{1-\beta},\ c + \beta \sum_{w \in W} f(w) v_{old}(w)\right\}$
    end
    error $= \max |v_{new}(w) - v_{old}(w)|$
    $v_{old}(w) = v_{new}(w)$
end
Return $v(w)$ and policy functions $e(w)$

References

Ljungqvist, L., & Sargent, T. J. (2018). *Recursive macroeconomic theory* (Fourth ed.). Cambridge, MA, USA: MIT Press.

McCall, J. J. (1970). Economics of information and job search. *Quarterly Journal of Economics*, *84*(1), 113 – 126.

[2] I thank Stefano Lord-Medrano for sharing the MATLAB code for solving the model.