ECON 714A: Macroeconomic Theory - Handout 6

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1 Consumption-Savings Choice Model from LS Chapter 17

1.1 Recap from Handout 4

We consider a consumer maximizing his lifetime utility in an incomplete markets economy.[1] The consumer can purchase only a state non-contingent claim. The agent chooses consumption c_t and saving a_{t+1} each period where the rate of return for this asset is given as r that satisfies $\beta(1+r) = 1$.

$$\max_{c_t, a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } a_{t+1} = (1+r)(a_t - c_t) + y_{t+1}$$
$$0 \leq c_t \leq a_t, \; a_0 \text{ given.}$$

We can write this problem as a recursive problem with the Bellman equation as below. Note that $V(a)$ is increasing, strictly concave, and differentiable in a.

$$V(a) = \max_{0 \leq c \leq a} u(c) + \beta E V\left((1+r)(a-c) + y\right)$$

We also denote b_t as the amount of one-period debt of the consumer: $b_t = y_t - a_t$ with $b_0 = 0$. Using this notation, the sequential budget constraint changes to $c_t + b_t \leq \beta b_{t+1} + y_t$.

1.2 Quadratic Preferences

We can solve for the linear quadratic permanent income model as a benchmark. We solve the sequential problem with budget constraint $c_t + b_t \leq \beta b_{t+1} + y_t$ and assume $E_0 \lim_{t\to\infty} \beta^t b_t^2 = 0$. We impose such assumption to rule out the Ponzi scheme. With this constraint, we can iterate the sequential budget constraint forward to obtain $b_t = E_t \sum_{j=0}^{\infty} \beta^j (y_{t+j} - c_{t+j})$ as in the natural debt limit section.

Now let $u(c_t) = -0.5(c_t - \gamma)^2$ with $\gamma > 0$ is a bliss consumption level. Marginal utility $u'(c_t) = \gamma - c_t$. We also allow for "negative" consumption and assume that the endowment process $\{y_t\}$ is a stationary stochastic process. From the household's optimal condition, we have that $u'(c_t) =$

[1] Please refer to Sections 17.4-6 of Ljungqvist and Sargent (2018) textbook.

$E_t u'(c_{t+1})$ and hence, $c_t = E_t c_{t+1}$. In other words, c_t is a martingale. We can also characterize the debt b_t as follows.

$$\begin{aligned} b_t &= E_t \sum_{j=0}^{\infty} \beta^j y_{t+j} - \frac{c_t}{1-\beta} \\ \iff c_t &= \frac{r}{1+r} \left[-b_t + E_t \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} \right] \end{aligned}$$

The equation above states that the consumption is equal to the annuity return for one's net wealth at the beginning of period t. This is another version of the permanent income hypothesis. Plug this into the sequential budget constraint and we have the following.

$$\begin{aligned} b_{t+1} &= (1+r)b_t + r \left[-b_t + E_t \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} \right] - (1+r)y_t \\ &= b_t + rE_t \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} - (1+r)y_t \end{aligned}$$

This implies that b_t has a unit root. It also follows from $E_t c_{t+1} = c_t$ that c_t has a unit root. The two processes are cointegrated as well. Because b_t and c_t are random-walk processes, they do not converge to certain level as $t \to \infty$, which is different from what we showed last time.

Now define the permanent component of income $y_{pt} \equiv \frac{r}{1+r} E_t \sum_{j=0}^{\infty} (1+r)^{-j} y_{t+j}$, which is the annuity return from present discounted value of future endowments. Then we have

$$b_{t+1} = b_t + (1+r)(y_{pt} - y_t)$$

Note that consumption c_t only depends on the expected value of present discounted value of the endowment process. Higher-order moments do not affect the consumption.

1.3 Stochastic Endowment Process IID

We will now consider an alternative specification of $u(c_t)$ where we rule out "negative" consumption. Going back to the recursive form of the consumer's problem, we can obtain the first-order condition as follows. Denote Π_s as the conditional probability of state s being realized next period, and s corresponds to the realization of an iid income $\bar{y}_s$.

$$u'(c) \geq \beta(1+r) \sum_s \Pi_s V'((1+r)(a-c) + \bar{y}_s)$$

This holds in equality with $c \leq a$ not binding. The envelope condition also yields $u'(c) = V'(a)$, so we can rewrite this condition as follows. $V(\cdot)$ is increasing, strictly concave, and differentiable.

$$V'(a) \geq \sum_s \Pi_s V'(a'_s)$$

Hence, $V'(a)$ is a non-negative supermartingale. By the supermartingale convergence theorem, $V'(a)$ converges almost surely. This means that with any generated value of y_t, $V'(a_t)$ converges to a single value. In particular, $V'(a_t)$ converges to zero. Suppose not, and $V'(a_t)$ converges to a finite positive value. Then this means that the asset a_t converges to a finite value, but this contradicts the iid nature of income process. Since asset a_t cannot converge, and $V'(a_t)$ converges to zero, which corresponds to $a_t \to \infty$.

Note that the asset sequence is not monotonically increasing even though it diverges to infinity. This is because the consumer uses his asset to self-insure to the income shocks. Formally, suppose that even the lowest income realization $\bar{y}_1$ is associated with nondecreasing assets, so $(1+r)(a-c)+\bar{y}_1 \geq a$.

$$V'((1+r)(a-c)+\bar{y}_1) \leq V'(a) = \sum_s \Pi_s V'(a'_s)$$

However, we have $V'(a'_s) \leq V'(a'_1)$ by concavity, so this is a contradiction. Hence, the asset sequence should decrease with low realizations of income shock.

We also have that the consumption sequence also diverges to infinity. By envelope condition, we have the following from the first order condition.

$$u'(c) \geq \sum_s \Pi_s u'(c'_s)$$

Suppose the consumption sequence converges to a finite limit. Then the maximum sustainable level of consumption $\bar{c}$ is the annuity return of present discounted value of future incomes $\{y_t\}$ and current asset a_t. However, if there is a high income realization at some time period, the consumer can increase his utility by consuming slightly more than what was planned in that period. Hence, such consumption level $\bar{c}$ is never optimal. This also implies that $u'(c)$ almost surely converges to 0 by the supermartinagle convergence theorem, because c diverges to infinity.

1.4 Stochastic Endowment Process in General

If we depart from the iid assumption of future incomes, the first-order condition changes to $u'(c_t) \geq E_t\left[u'(c_{t+1})\right]$. Assuming that the utility function is bounded, there is a proposition by Chamberlain and Wilson (2000) with an arbitrary stationary endowment process.

Proposition. If there is an $\epsilon > 0$ such that for any $\alpha \in \mathbb{R}^+$ and any information set $I_t, \forall t \geq 0$,

$$\Pr\left(\alpha \leq \frac{r}{1+r}\sum_{j=t}^{\infty}(1+r)^{t-j}y_j \leq \alpha+\epsilon \,\middle|\, I_t\right) < 1-\epsilon$$

then $\Pr(\lim_{t\to\infty} c_t = \infty) = 1$.

This proposition states that as long as the annuity return of future incomes is sufficiently stochastic, then the consumption sequence will never converge. This contrasts with the result in the deterministic case where we showed that the consumption will converge to the supremum of the annuity return of future incomes.

2 Incomplete Markets Economy on Huggett (1993)

This is a summary of the model from Dean Corbae's lecture note for Huggett (1993) economy in the Econ 899 course. You can also find Dean's lecture note on his website.

2.1 Environment

There are unit measure of agents who maximize $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ where u is continuously differentiable, strictly increasing, strictly concave, and bounded. Agents face two earning shocks $s \in \{e, u\}$ that follow a Markov chain with transition probability $\pi(s'|s)$. The shock is iid across agents, and state e gives income of 1 while u gives $b < 1$. Agents can buy one-period state non-contingent bonds at a discount price q and they are subject to borrowing constraint $\tilde{a} \leq 0$.

Agents have different histories of s, so they will hold different amount of assets. Hence, denote μ as the distribution of agents over the space of states $s \in S$ and assets $a \in A$ given the bond price q. So μ is a probability measure defined on a product space of $S \times A$.

2.2 Recursive Equilibrium

The recursive steady state equilibrium is an allocation (c, a'), price q, and a stationary distribution μ such that

1. For a given level of q, (c, a') solves the household's problem.

$$
\begin{aligned}
v(s, a) &= \max_{a'} \; u(y(s) + a - qa') + \beta \sum_{s'} \pi(s'|s) v(s', a') \\
\text{s.t. } \tilde{a} &\leq a \leq \frac{y(s) + a}{q}
\end{aligned}
$$

2. Given μ, goods and asset markets clear.

$$
\begin{aligned}
\int_{S,A} c(s, a; q) - y(s) \, d\mu &= 0 \\
\int_{S,A} a'(s, a; q) d\mu &= 0
\end{aligned}
$$

3. μ is a stationary distribution. For any $S_0 \subseteq S$, $A_0 \subseteq A$, μ satisfies the following.

$$
\mu(S_0, A_0; q) = \int_{S_0, A_0} \left[\int_{S,A} \mathbb{I}(a'(a, s; q) \in A_0) \pi(s'|s) d\mu \right] ds' da'
$$

Hopefully, you will learn more mathematical and computational details in Econ 899: Computation course next year:)

Appendix: Supermartingale Convergence Theorem from Ljungqvist and Sargent (2018), p.779

Let the elements of the 3-tuple $(\Omega, \mathcal{F}, P)$ denote a sample space, a collection of events, and a probability measure, respectively. Let $t \in T$ index time, where T denotes the nonnegative integers. Let $\mathcal{F}_t$ denote an increasing sequence of σ-fields of $\mathcal{F}$ sets. Suppose that

(i) Z_t is measurable with respect to $\mathcal{F}_t$;

(ii) $E\,|Z_t| < +\infty$;

(iii) $E\,(Z_t \mid \mathcal{F}_s) = Z_s$ almost surely for all $s < t; s, t \in T$.

Then $\{Z_t, t \in T\}$ is said to be a martingale with respect to $\mathcal{F}_t$. If (iii) is replaced by $E\,(Z_t \mid \mathcal{F}_s) \geq Z_s$ almost surely, then $\{Z_t\}$ is said to be a submartingale. If (iii) is replaced by $E\,(Z_t \mid \mathcal{F}_s) \leq Z_s$ almost surely, then $\{Z_t\}$ is said to be a supermartingale.

Supermartingale Convergence Theorem: Let $\{Z_t, \mathcal{F}_t\}$ be a nonnegative supermartingale. Then there exists a random variable Z such that $\lim Z_t = Z$ almost surely and $E|Z| < +\infty$, i.e., Z_t converges almost surely to a finite limit.

References

Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, *17*(5), 953-969.

Ljungqvist, L., & Sargent, T. J. (2018). *Recursive macroeconomic theory* (Fourth ed.). Cambridge, MA, USA: MIT Press.