Note 3: Jovanovic Model of Job Matching *

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1 Environment

We consider a model of matching between workers and firms in the labor market following Jovanovic (1979). This model focuses on explaining three facts: (1) wages tend to rise with tenure on the job, (2) job quits are negatively correlated with tenure, and (3) the probability of a subsequent quit is negatively correlated with current wage. To address these issues, the model considers an economy in which each worker and a firm form a match with match quality θ drawn from a distribution. The model is an infinite-horizon model and each matching process has three stages across two periods.

In the first stage, an unemployed worker has not chosen a match parameter but it has expected present value of Q before drawing. In the second stage, the worker draws a match parameter $\theta \sim N(\mu, \sigma_0^2)$ with an orthogonal error $u \sim N(0, \sigma_u^2)$. Hence, what he observes is $y = \theta + u$ and the firm knows this so it offers $E[\theta|y]$ for this period and θ in the following future periods to the worker. If the worker chooses to accept the offer, the match proceeds to the third stage when the two parties get to know the exact value of θ. If it refuses the offer, then it will draw a new match parameter and noise next period. Then in the third stage, both parties know the exact value of θ and the worker has to choose whether to accept to work with receiving θ afterwards or not. We can solve this problem using backward induction.

Before diving into the backward induction the worker's problem, we briefly go over the conditional distribution of expected wage offers that the workers will face in each period. The idea is related to Kalman filter.

2 Conditional Distributions of θ and $E[\theta|\theta + u]$

We first begin with the distribution of θ conditional to y. This is the distribution that the worker faces in the second stage after drawing a noisy observation of θ. We can think of this similar to regressing θ by $y = \theta + u$ as in linear regression. ε below is a mean-zero error orthogonal to y and we use the covariance independence of θ and y to obtain $E(\theta|y)$.

*This note follows Section 6.8 of Ljungqvist and Sargent (2018).

Assuming that $E(\theta|y)$ is linear in y, we can obtain the following. We also know that the unconditional mean of θ and y are both μ. We use the fact that $E\left[E(\theta|y)\right] = \mu$ to derive a.

$$\begin{aligned}
\theta &= E(\theta|y) + \varepsilon = a + by + \varepsilon \\
Cov(\theta - E(\theta|y), y) &= 0 \\
\iff Cov(\theta, y) &= Cov(E(\theta|y), y) \\
&= E(a + by - \mu)(y - \mu) \\
&= bE(y-\mu)^2 + b\mu E(y-\mu) = bE(y-\mu)^2 \\
\therefore b &= \frac{Cov(\theta, y)}{Var(y)} = \frac{\sigma_0^2}{\sigma_0^2 + \sigma_u^2} \\
a &= \left(1 - \frac{\sigma_0^2}{\sigma_0^2 + \sigma_u^2}\right)\mu
\end{aligned}$$

We now denote $m_0 \equiv E(\theta|y)$. We can also obtain the conditional variance of θ using that it is just the unconditional variance of ε.

$$\begin{aligned}
\sigma_1^2 \equiv Var(\theta|y) &= Var(\varepsilon) \\
&= Var(\theta) - Var(E(\theta|y)) \\
&= \sigma_0^2 - \left(\frac{\sigma_0^2}{\sigma_0^2 + \sigma_u^2}\right)^2 (\sigma_0^2 + \sigma_u^2) \\
&= \frac{\sigma_0^2}{\sigma_0^2 + \sigma_u^2}\sigma_u^2
\end{aligned}$$

We have assumed that θ, u, and y are all normal distributed. Hence, we obtain that $\theta|y \sim N(m_0, \sigma_1^2)$. Also we can characterize the distribution of $E(\theta|y)$ from the normality of y. We have obtained the variance of $E(\theta|y)$ in the third equality above, so we have the following.

$$E(\theta|y) = m_0 \sim N\left(\mu, \frac{\sigma_0^4}{\sigma_0^2 + \sigma_u^2}\right)$$

3 Recursive Formation and Solution

3.1 Third Stage

We proceed to the recursive formation and solution of the worker's problem. In the third stage, worker knows θ and has to choose whether or not to work this period and forever more at the wage level θ. Denote $J(\theta)$ as the expected present value of perpetual flow of income at this stage. If he accepts the offer, he attains $\theta + \beta J(\theta)$. If he refuses the offer, he has to draw another offer with a noisy error next period which has the expected present value of Q. Hence the Bellman equation can be written as $J(\theta) = \max\{\theta + \beta J(\theta), \beta Q\}$.

Note that reservation wage policy is the optimal policy as well, because $J(\theta) = \frac{\theta}{1-\beta}$ once it is larger than βQ which is a constant. Hence, the left term in the parenthesis of $\max$ operator is an increasing function of θ while the right term is a constant, so the two functions coincide at a single value of $\bar{\theta}$. This is the reservation wage level in which the worker is indifferent between accepting and refusing the offer. Hence we have the following optimal rule.

$$
\begin{aligned}
J(\theta) &= \begin{cases} \theta + \beta J(\theta) = \frac{\theta}{1-\beta} & \text{if } \theta \geq \bar{\theta} \\ \beta Q & \text{if } \theta < \bar{\theta} \end{cases} \\
\frac{\bar{\theta}}{1-\beta} &= \beta Q
\end{aligned}
$$

3.2 Second Stage

In the second stage, worker draws a noisy signal $y = \theta + u$ and the firm offers $m_0 = E(\theta|y)$ to the worker. Let $V(m_0)$ be the expected present value of wages of a worker who has an offer m_0 at the second stage. If he accepts the offer, he will get m_0 this period and get the offer of the real θ' in the next stage which has a value $J(\theta')$. Note that the θ' realized next period follows a conditional distribution of $N(m_0, \sigma_1^2)$ with distribution function $F(\theta'|m_0, \sigma_1^2)$ that we derived previously. If he refuses the offer, he will draw another offer next period that has an expected present value of Q.

Bellman equation in the second period can be written as follows.

$$
\begin{aligned}
V(m_0) &= \max \left\{ m_0 + \beta \int J(\theta') dF(\theta'|m_0, \sigma_1^2),\ \beta Q \right\} \\
&= \begin{cases} m_0 + \beta \int J(\theta') dF(\theta'|m_0, \sigma_1^2) & \text{if } m_0 \geq \bar{m}_0 \\ \beta Q & \text{if } m_0 < \bar{m}_0 \end{cases}
\end{aligned}
$$

Since $J(\theta)$ is a weakly increasing function of θ, the integral sum will sum up higher values of $J(\theta')$ with higher m_0. Hence, the left term in the parenthesis is increasing in m_0. Again, βQ is a constant here. So we can characterize a reservation wage policy at $\bar{m}_0$ as the optimal policy here. Hence, $\bar{m}_0$ satisfies the following equation.

$$
V(\bar{m}_0) = \bar{m}_0 + \beta \int J(\theta') dF(\theta'|\bar{m}_0, \sigma_1^2) = \beta Q
$$

Plugging in the functional form of $J(\theta')$, we can derive that the reservation wage $\bar{\theta}$ at stage 3 is higher than $\bar{m}_0$.

$$
\begin{aligned}
V(\bar{m}_0) &= \bar{m}_0 + \beta \frac{\bar{\theta}}{1-\beta} \int_{-\infty}^{\bar{\theta}} dF(\theta'|\bar{m}_0, \sigma_1^2) + \frac{\beta}{1-\beta} \int_{\bar{\theta}}^{\infty} \theta' dF(\theta'|\bar{m}_0, \sigma_1^2) \\
&= \frac{\bar{\theta}}{1-\beta} = \frac{\bar{\theta}}{1-\beta} \int_{-\infty}^{\bar{\theta}} dF(\theta'|\bar{m}_0, \sigma_1^2) + \frac{\bar{\theta}}{1-\beta} \int_{\bar{\theta}}^{\infty} dF(\theta'|\bar{m}_0, \sigma_1^2) \\
&\Rightarrow \frac{\bar{\theta}}{1-\beta} \int_{-\infty}^{\bar{\theta}} dF(\theta'|\bar{m}_0, \sigma_1^2) - \beta \frac{\bar{\theta}}{1-\beta} \int_{-\infty}^{\bar{\theta}} dF(\theta'|\bar{m}_0, \sigma_1^2) - \bar{m}_0 \\
&= \frac{\beta}{1-\beta} \int_{\bar{\theta}}^{\infty} \theta' dF(\theta'|\bar{m}_0, \sigma_1^2) - \frac{\bar{\theta}}{1-\beta} \int_{\bar{\theta}}^{\infty} dF(\theta'|\bar{m}_0, \sigma_1^2)
\end{aligned}
$$

From the last equation, we have the following.

$$
\begin{aligned}
\bar{\theta}\int_{-\infty}^{\bar{\theta}} dF(\theta'|\bar{m}_0,\sigma_1^2) - \bar{m}_0 &= \frac{1}{1-\beta}\int_{\bar{\theta}}^{\infty}(\beta\theta' - \bar{\theta})dF(\theta'|\bar{m}_0,\sigma_1^2) \\
\bar{\theta} - \bar{m}_0 &= \int_{\bar{\theta}}^{\infty}\left(\frac{\beta\theta'}{1-\beta} - \frac{\bar{\theta}}{1-\beta} + \bar{\theta}\right)dF(\theta'|\bar{m}_0,\sigma_1^2) \\
&= \frac{\beta}{1-\beta}\int_{\bar{\theta}}^{\infty}\left(\theta' - \bar{\theta}\right)dF(\theta'|\bar{m}_0,\sigma_1^2) > 0
\end{aligned}
\tag{1}
$$

Hence, we have that $\bar{\theta} > \bar{m}_0$ so the reservation wage rises from stage 2 to stage 3. One interpretation of equation (1) is that the opportunity cost of working this period with a wage lower than $\bar{\theta}$, which is the left hand side, should be equal to the expected benefit of earning some wage higher than $\bar{\theta}$ starting the next period at stage 3, which is the right hand side. Hence if a worker draws gets an offer $m_0 \in (\bar{m}_0, \bar{\theta})$ at stage 2, he will accept the offer and work, speculating a higher value of θ to be realized next period in stage 3.

3.3 First Stage

In the first stage, the agent has yet to draw the noisy observation of θ and observes Q as the expected value of wages that he will draw. Q is defined as an expectation over the distribution of m_0, the wage offer that the agent will get in stage 2.

$$
Q = \int V(m_0)dG\left(m_0|\mu, \frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)
$$

3.4 Characterization of the Solution

We need to characterize the value function $V(m_0)$ in the second stage to fully characterize functions $J(\theta)$ and Q and solve the model. We can find V from the Bellman equation in Section 3.2. Plugging in the functions J and Q, we have the following.

$$
V(m_0) = \max\left\{m_0 + \beta\int \max\left[\frac{\theta}{1-\beta}, \beta\int V(m_1^{'})dG(m_1^{'}|\cdot)\right]dF(\theta|m_0,\sigma_1^2), \beta\int V(m_1^{'})dG(m_1^{'}|\cdot)\right\}
$$

Define the right hand side of the equation as a functional operator T on V, then we can show that T is a contraction.

1. Monotonicity: If $V(m_0) \geq Z(m_0), \forall m_0$, then we have the following.

$$
\begin{gathered}
\beta\int V(m_1^{'})dG(m_1^{'}|\cdot) \geq \beta\int Z(m_1^{'})dG(m_1^{'}|\cdot) \\
\max\left[\frac{\theta}{1-\beta}, \beta\int V(m_1^{'})dG(m_1^{'}|\cdot)\right] \geq \max\left[\frac{\theta}{1-\beta}, \beta\int Z(m_1^{'})dG(m_1^{'}|\cdot)\right], \forall\theta \\
\therefore TV(m_0) \geq TZ(m_0)
\end{gathered}
$$

2. Discounting: For all positive $c > 0$, we have the following.

$$
\begin{aligned}
T(V+c)(m_0) = \max\Bigg\{ & m_0 + \beta \int \max\left[\frac{\theta}{1-\beta}, \beta \int V(m_1^{'})dG(m_1^{'}|\cdot) + \beta c\right] dF(\theta|m_0, \sigma_1^2), \\
& \beta \int V(m_1^{'})dG(m_1^{'}|\cdot) + \beta c\Bigg\} \\
& = \max\Bigg\{ m_0 + \beta \int_{-\infty}^{\tilde{\theta}} \beta \int V(m_1^{'})dG(m_1^{'}|\cdot)dF(\theta|m_0, \sigma_1^2) \\
& + \beta \int_{\tilde{\theta}}^{\infty} \frac{\theta}{1-\beta} dF(\theta|m_0, \sigma_1^2) + \beta^2 c F(\tilde{\theta}|m_0, \sigma_1^2), \\
& \beta \int V(m_1^{'})dG(m_1^{'}|\cdot) + \beta c\Bigg\} \\
& \le \max\Bigg\{ m_0 + \beta \int \max\left[\frac{\theta}{1-\beta}, \beta \int V(m_1^{'})dG(m_1^{'}|\cdot)\right] dF(\theta|m_0, \sigma_1^2) + \beta^2 c, \\
& \beta \int V(m_1^{'})dG(m_1^{'}|\cdot) + \beta c\Bigg\} \\
& \le \max\Bigg\{ m_0 + \beta \int \max\left[\frac{\theta}{1-\beta}, \beta \int V(m_1^{'})dG(m_1^{'}|\cdot)\right] dF(\theta|m_0, \sigma_1^2) + \beta c, \\
& \beta \int V(m_1^{'})dG(m_1^{'}|\cdot) + \beta c\Bigg\} \\
& = TV(m_0) + \beta c
\end{aligned}
$$

$\tilde{\theta}$ denotes the threshold level of θ where $\frac{\theta}{1-\beta} = \beta \int V(m_1^{'})dG(m_1^{'}|\cdot) + \beta c$.

Since T is a contraction, we can find V as the fixed point of T and solve the model.

4 Endogenous Statistics

The model proceeds to explain three facts: (1) wages tend to rise with tenure on the job, (2) job quits are negatively correlated with tenure, and (3) the probability of a subsequent quit is negatively correlated with current wage.

4.1 Wage rises with tenure

Probability of a worker working in the second stage is Prob$(m_0 \ge \bar{m}_0)$. The mean wage of those employed in this period can be written as

$$
\bar{w}_1 = \frac{\int_{\bar{m}_0}^{\infty} m_0 dG\left(m_0|\mu, \frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}{1 - G\left(\bar{m}_0|\mu, \frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}
$$

Probability of a worker continuing to work in the third stage is Prob$(m_0 \ge \bar{m}_0$ and $\theta \ge \bar{\theta})$. From this, the mean wage of those working in the second period of their tenure (third stage) is obtained as

$$\bar{w}_2 = \frac{\int_{\bar{m}_0}^{\infty}\int_{\bar{\theta}}^{\infty}\theta dF(\theta|m_0,\sigma_1^2)dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}{\int_{\bar{m}_0}^{\infty}(1-F(\bar{\theta}|m_0,\sigma_1^2))dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}$$

From this we obtain $\bar{w}_1 < \bar{w}_2$.

$$\begin{aligned}
\bar{w}_1 &= \frac{\int_{\bar{m}_0}^{\infty} m_0 dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}{1-G\left(\bar{m}_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)} \\
&= \frac{1}{1-G\left(\bar{m}_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}\left[\int_{\bar{m}_0}^{\infty}\int_{-\infty}^{\bar{\theta}}\theta dF(\theta|m_0,\sigma_1^2)dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)\right. \\
&\left.+\int_{\bar{m}_0}^{\infty}\int_{\bar{\theta}}^{\infty}\theta dF(\theta|m_0,\sigma_1^2)dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)\right] \\
&= \frac{1}{1-G\left(\bar{m}_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}\left[\int_{\bar{m}_0}^{\infty}\int_{-\infty}^{\bar{\theta}}\theta dF(\theta|m_0,\sigma_1^2)dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)\right. \\
&\left.+\bar{w}_2\int_{\bar{m}_0}^{\infty}(1-F(\bar{\theta}|m_0,\sigma_1^2))dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)\right] \\
&< \frac{1}{1-G\left(\bar{m}_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}\int_{\bar{m}_0}^{\infty}\left\{\bar{\theta}F(\bar{\theta}|m_0,\sigma_1^2)+\bar{w}_2(1-F(\bar{\theta}|m_0,\sigma_1^2))\right\}dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right) \\
&= \frac{1}{1-G\left(\bar{m}_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}\int_{\bar{m}_0}^{\infty}\left\{\bar{w}_2-(\bar{w}_2-\bar{\theta})F(\bar{\theta}|m_0,\sigma_1^2)\right\}dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right) \\
&< \frac{1}{1-G\left(\bar{m}_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right)}\int_{\bar{m}_0}^{\infty}\bar{w}_2 dG\left(m_0|\mu,\frac{\sigma_0^4}{\sigma_0^2+\sigma_u^2}\right) = \bar{w}_2
\end{aligned}$$

The last inequality follows from $\bar{w}_2 > \bar{\theta}$. Hence the mean wage of workers in the second period of tenure is higher than that of workers in the first period of tenure. Hence, $\bar{w}_1 < \bar{w}_2$ and $\bar{m}_0 < \bar{\theta}$ together implies that wages increase with tenure.

4.2 Job quits decrease with tenure

By construction, job quits only occur between the first and the second periods of tenure just before the realization of θ in stage 3. After then, there is no job quit. Hence, job quits decrease with tenure.

4.3 Subsequent quits are fewer for higher wages

The probability of a subsequent quit in stage 3 is Prob$(\theta < \bar{\theta}|m_0)$, which is the case when the realized θ is smaller than the reservation wage $\bar{\theta}$. This probability is decreasing in m_0, the conditional mean of θ given the noisy observation, by construction.

References

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